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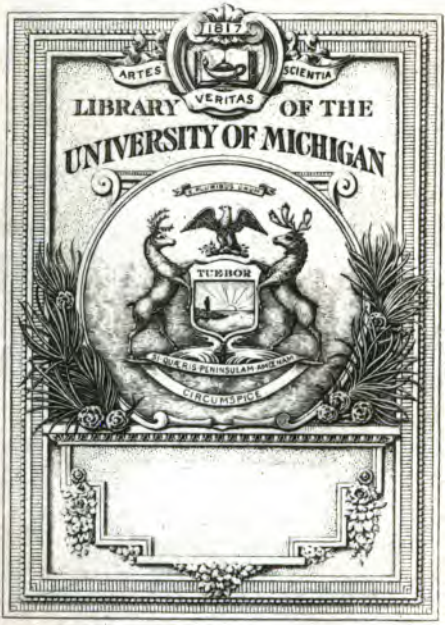

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THE

MATHEMATICAL

REPOSITORY.

BY T. LEYBOURN,

OF THE ROYAL MILITARY COLLEGE.

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THE

MATHEMATICAL REPOSITORY.

ARTICLE I.

An Inveſtigation of Theorems relating to the ſums of the Powers of perpendiculars to all the ſides of a regular polygon, from any point in the circumference of its inſcribed circle.

By Mr. JAMES CUNLIFFE, *Bolton-le-Moors.*

ARTICLE 1. LEMMA. To find the ſums of the various powers of the roots of the equation

$v^n - Pv^{n-1} + Qv^{n-2} - Rv^{n-3} + Sv^{n-4}$ &c. $= 0$; n being a whole number.

Let the roots of the equation be denoted by a, b, c, d &c. the number of which is n, as is very well known: Then by the property of equations

$v^n - Pv^{n-1} + Qv^{n-2} - Rv^{n-3} + Sv^{n-4}$ &c. $= (v-a)(v-b)(v-c)(v-d)$ &c. to n factors.

By taking the fluxions of the hyperbolic logarithms of each ſide of the equation, and dividing by v there will be had

$$\frac{nv^{n-1} - (n-1)Pv^{n-2} + (n-2)Qv^{n-3} - (n-3)Rv^{n-4} + (n-4)Sv^{n-5} \text{ &c.}}{v^n - Pv^{n-1} + Qv^{n-2} - Rv^{n-3} + Sv^{n-4} \text{ &c.}}$$

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$$= \left\{ \begin{matrix} \frac{1}{v-a} + \frac{1}{v-b} + \frac{1}{v-c} + \frac{1}{v-d} \text{ \&c.} \\ \text{to } n \text{ terms} \end{matrix} \right\} = \begin{cases} \frac{1}{v} + \frac{a}{v^2} + \frac{a^2}{v^3} + \frac{a^3}{v^4} + \frac{a^4}{v^5} \text{ \&c.} \\ \frac{1}{v} + \frac{b}{v^2} + \frac{b^2}{v^3} + \frac{b^3}{v^4} + \frac{b^4}{v^5} \text{ \&c.} \\ \frac{1}{v} + \frac{c}{v^2} + \frac{c^2}{v^3} + \frac{c^3}{v^4} + \frac{c^4}{v^5} \text{ \&c.} \\ \frac{1}{v} + \frac{d}{v^2} + \frac{d^2}{v^3} + \frac{d^3}{v^4} + \frac{d^4}{v^5} \text{ \&c.} \end{cases}$$

by actually expanding the fractions $\frac{1}{v-a}$, $\frac{1}{v-b}$ &c. where it is evident that the vertical columns must be continued to n terms.

Put $A = a + b + c + d$ &c. to n terms
$B = a^2 + b^2 + c^2 + d^2$ &c. to n terms
$C = a^3 + b^3 + c^3 + d^3$ &c. to n terms
$D = a^4 + b^4 + c^4 + d^4$ &c. to n terms. Then will

$$\frac{nv^{n-1} - (n-1)Pv^{n-2} + (n-2)Qv^{n-3} - (n-3)Rv^{n-4} + (n-4)Sv^{n-5} \text{ \&c.}}{v^n - Pv^{n-1} + Qv^{n-2} - Rv^{n-3} + Sv^{n-4} \text{ \&c.}}$$

$$= \frac{n}{v} + \frac{A}{v^2} + \frac{B}{v^3} + \frac{C}{v^4} + \frac{D}{v^5} \text{ \&c.}$$ and by multiplying by

$v^n - Pv^{n-1} + Qv^{n-2} - Rv^{n-3} + Sv^{n-4}$ &c. there will be had

$$\begin{aligned} & nv^{n-1} - (n-1)Pv^{n-2} + (n-2)Qv^{n-3} - (n-3)Rv^{n-4} + (n-4)Sv^{n-5} \text{ \&c.} \\ = \; & nv^{n-1} + Av^{n-2} + Bv^{n-3} + Cv^{n-4} + Dv^{n-5} \text{ \&c.} \\ & - nPv^{n-2} - PAv^{n-3} - PBv^{n-4} - PCv^{n-5} \text{ \&c.} \\ & + nQv^{n-3} + QAv^{n-4} + QBv^{n-5} \text{ \&c.} \\ & - nRv^{n-4} - RAv^{n-5} \text{ \&c.} \\ & + nSv^{n-5} \text{ \&c.} \end{aligned}$$

Equating the co-efficients of the homologus terms, there will be had, $-(n-1)P = A - nP$; $(n-2)Q = B - PA + nQ$; $(n-3)R = C - PB + QA - nR$; $(n-4)S = D - PC + QB - RA + nS$, &c. From the firſt of theſe equations $A = P$; from the ſecond $B = PA - 2Q = P^2 - 2Q$; from the third $C = PB - QA + 3R = P^3 - 3PQ + 3R$; and from the fourth $D = PC - QB + RA - 4S = P^4 - 4P^2Q + 2Q^2 + 4PR - 4S$, &c. &c.

It

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It may be remarked, by the way, from what is just deduced, that if, in any equation, the indices of the unknown quantity are whole numbers, and the co-efficients of the several terms rational; then the sums of the various powers of the roots will be rational also. The truth of this remark is evident from barely inspecting the formulæ themselves; for nothing is to be found in the formulæ but rational quantities. Therefore, if imaginary, or surd quantities are any ways concerned in the roots themselves, the affirmative and negative parts of such imaginary or surd quantities, which are concerned in the sums of the various powers of the roots must mutually destroy each other.

Art. 2. If v be the versed sine of a circular arc A, whose radius is r; and V the versed sine of any multiple n of that arc or of nA; then per Cor. 2, Prop. 29, Book I. of Emerson's Trigonometry,

$$\frac{(2v)^n}{2r^{n-1}} - \frac{2nrA}{2v} - \frac{(2n-3)rB}{2.2v} - \frac{(2n-4)(2n-5)rC}{3(2n-3).2v} - \frac{(2n-5)(2n-7)rD}{4(2n-4).2v}$$

&c. $= \pm V$; where A, B, C, D &c. denote the preceding terms with their signs: therefore by restoring the values of A, B, C, D, &c. and multiplying the resulting expression by $\frac{r^{n-1}}{2^{n-1}}$ it becomes

$$\left.\begin{aligned} v^n - nrv^{n-1} + \frac{n(2n-3)r^2v^{n-2}}{2^2} - \frac{n(2n-4)(2n-5)r^3v^{n-3}}{2^3.3} \\ + \frac{n(2n-5)(2n-6)(2n-7)r^4v^{n-4}}{2^4.3.4} \text{ &c.} \end{aligned}\right\} = \frac{r^{n-1}}{2^{n-1}} \times \pm V.$$

The former of the preceding equations will also express the versed sine of the arcs $C + nA$, $2C + nA$, $3C + nA$, $4C + nA$ &c. where C denotes the whole periphery of the circle. Therefore the several values of v, or roots of the preceding equation will be the versed sines of the arcs, A, $\frac{C}{n} + A$, $\frac{2C}{n} + A$, $\frac{3C}{n} + A$, $\frac{4C}{n} + A$ &c. continued to n terms.

Art. 3. Fig. 496, pl. 27. If a regular polygon the number of whole sides is n, be circumscribed about a circle; M', M'', M''',

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$M''',\ldots\ldots M^{n}$ being the points of contact with the sides; and if from any point P, in the arc $M'M^{n}$ perpendiculars PQ, PR, PS, &c. be drawn to the sides M', M'', M''' &c. these lines will be respectively equal to the versed sines of the arcs PM', PM'', PM''' &c.

Demonstration. Through the point of contact M', draw the diameter M'N and P*n* perpendicular to it. Then QM'*n* is a right angle (Eu. 18. III.), wherefore the figure QM'*n*P is a rectangle, and (Eu. 34. I.), PQ = M'*n*, the versed sine of the arc PM'. And in the same manner it may be demonstrated that PR is equal to the versed sine of the arc PM'' &c.

Now if the diameter M'N = 2*r*, C = the whole periphery of the circle, and arc PM' = A; then will $\frac{C}{n}$ be equal to each of the arcs $M^{n}M'$, M'M'', M''M''' &c. therefore $PM'' = \frac{C}{n} + A$, $PM''' = \frac{2C}{n} + A$, &c. and the perpendiculars PQ, PR, PS, &c. being respectively equal to the versed sines of the arcs A, $\frac{C}{n} + A$, $\frac{2C}{n} + A$ &c. will manifestly be the roots of the equation in Art. 2.

Art 4. It is required to find the sums of the various powers of the perpendiculars PQ, PR, PS, &c.

By comparing the co-efficients of the homologous terms of the equations in Article 1 and 2, there will be had $P = nr$; $Q = \frac{n(2n-3)r^{2}}{2^{2}}$; $R = \frac{n(2n-4)(2n-5)r^{3}}{2^{3}\cdot 3}$; $S = \frac{n(2n-5)(2n-6)(2n-7)r^{4}}{2^{4}\cdot 3\cdot 4}$ &c.

Then $A = P = nr$ = to the sum of all the perpendiculars PQ, PR, PS, &c.

$B = PA - 2Q = n^{2}r^{2} - \frac{n(2n-3)r^{2}}{2} = \frac{3nr^{2}}{2}$ = to the sum of the squares of all the perpendiculars PQ, PR, PS, &c.

$C = PB - QA + 3R = \frac{3n^{2}r^{3}}{2} - \frac{n^{2}(2n-3)r^{3}}{2^{2}} + \frac{n(2n-4)(2n-5)r^{3}}{2^{3}} = \frac{5nr^{3}}{2}$ = to the sum of the cubes of all the perpendiculars PQ, PR, PS, &c.

D=

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$$D = PC - QB + RA - 4S = \frac{5n^2r^4}{2} - \frac{3n^3(2n-3)r^4}{2^3}$$
$$+ \frac{n^3(2n-4)(2n-5)r^4}{2^3 \cdot 3} - \frac{n(2n-5)(2n-6)(2n-7)r^4}{2^4 \cdot 3}$$
$$= \frac{35nr^4}{8} = \text{to the sum of the fourth powers of all the perpendiculars PQ, PR, \&c.}$$

And by proceeding in the same way we shall obtain $\frac{63nr^5}{8}$ for the sum of the fifth powers of the said perpendiculars.

The two formulæ for the sum of the squares and sum of the cubes of the perpendiculars PQ, PR, PS, &c. agree perfectly with what is demonstrated by Messrs. Lowry and Swale in the first volume of the Repository.

ARTICLE II.

Improved Solutions to some Curious Mathematical Problems.

PROB. I. *By Mr.* JAMES CUNLIFFE.

TO find three square numbers, such, that the difference of every two of them may be a square number.

SOLUTION.

Let the numbers be denoted by x^2, y^2, and z^2;
then by the question $x^2-y^2=a^2$, $x^2-z^2=b^2$, and $y^2-z^2=$ a square.
From the two first equations $x^2 = a^2 + y^2 = b^2 + z^2$.
Put $b = a + sv$, and $z = y = rv$. Then
$a^2+y^2=b^2+z^2=(a+sv)^2+(y-rv)^2=a^2+y^2+2sav-2ryv+s^2v^2$
$+ r^2v^2$; therefore,

$$v = \frac{2ry-2sa}{r^2+s^2}, \text{ and } z = y - rv = y - \frac{2r^2y-2rsa}{r^2+s^2} =$$
$$\frac{2rsa-y(r^2-s^2)}{r^2+s^2}.$$

But $x^2=a^2+y^2$, theref. put $(r^2+s^2)2mn=a$, and $(r^2+s^2)(m^2-n^2)$
$= y$, then will
$x=(r^2+s^2)(m^2+n^2)$, and $z=4rsmn-(r^2-s^2)(m^2-n^2)$. Wherefore

y^2-z^2

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$y^2-z^2 = (r^4+s^4)^2(m^2-n^2)^2 \cdots (r^2-s^2)(m^2-n^2)^2 + 8rs(r^2-s^2)mn$
$(m^2 \quad n^2) - 16r^2s^2m^2n^2$
$= 4r^2s^2(m^2-n^2)^2+8rs(r^2-s^2)mn(m^2-n^2)-16r^2s^2m^2n^2 =$
a ſquare by the queſtion. Aſſume $2rs(m^2+n^2)-$
$2mn\ (r^2-s^2)$ for its root, and then,
$4r^2s^2(m^2-n^2)^2+8rs(r^2-s^2)mn(m^2-n^2)-16r^2s^2m^2n^2=[2rs(m^2+n^2)$
$2m\ .\ (r^2-s^2)]^2$
$= 4r^2s^2(m^2+n^2)^2-8rs(r^2-s^2)mn(m^2+n^2)+(r^2-s^2)^2 4m^2n^2$, which being properly reduced gives $\frac{m}{n} = \frac{r^4+6r^2s^2+s^4}{4rs(r^2-s^2)}$; whence it appears that m and n may be expounded by any two numbers in the ratio of $r^4+6r^2s^2+s^4$ to $4rs\ (r^2-s^2)$ reſpectively, where r and s may be taken at pleaſure.

For example, if $r = 3$ and $s = 1$, then $\frac{m}{n} = \frac{136}{96} = \frac{17}{12}$; therefore put $m = 17$, and then $n = 12$, whence $x = 4330$, $y = 1450$, and $z = 1288$, which are the roots of three numbers with the required property. Or their halves, viz. 2165, 725, and 644 will be the roots of three ſquares that will anſwer.

Or the ſolution may be as follows.

Making uſe of the expreſſions for x, y, and z, which were obtained in the foregoing ſolution, it is plain that if we can determine $\frac{z+y}{2}$ and $\frac{z-y}{2}$, each equal to a rational ſquare, the queſtion will be anſwered, for then $\frac{z+y}{2} + \frac{z-y}{2} = \frac{z^2-y^2}{4}$ muſt be a rational ſquare, becauſe each of its factors is ſo. Now $\frac{z+y}{2} = s^2(m^2-n^2) + 2rsmn$, and $\frac{z-y}{2} = r^2(n^2-m^2) + 2rsmn$ are both to be ſquares; let the firſt of them be divided by s^2, and the other by r^2, and the quotients will be $m^2-n^2+\frac{2tmn}{s}$, and $n^2-m^2+\frac{2mn}{t}$, by putting $t = \frac{r}{s}$. Now theſe two laſt expreſſions muſt manifeſtly be ſtill ſquare numbers; therefore aſſume $qn-m$ for the root of the firſt of them, that is, put $m^2-n^2+2tmn=(qn-m)^2=q^2n^2-2qmn+m^2$, whence $n = 2m \times \frac{t+q}{q^2+1}$, which being written for n in the other expreſſion there will be had

n^2-m^2

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$n^2 - m^2 + (2nm \div t) = 4m^2(t+q)^2 \div (q^2+1)^2 - m^2 + 4m^2(t+q) \div t(q^2+1) =$
$m^2 \div t^2(q^2+1)^2 \times [4t^2(t+q)^2 - t^2(q^2+1)^2 + 4t(t+q)(q^2+1)] =$ a square, and therefore,
$4t^2(t+q)^2 - t^2(q^2+1)^2 + 4t(t+q)(q^2+1) = 4t^4 + 8t^3q + t^2(6q^2 - q^4 + 3) + 4tq(q^3+1)$ must be a square. Assume $2t^2 + 2tq$ for the root of this square, and then
$4t^4 + 8t^3q + t^2(6q^2 - q^4 + 3) + 4tq(q^3+1) = (2t^2+2tq)^2 = 4t^4 + 8t^3q + 4t^2q^2$, whence

$t = \frac{r}{s} = \frac{4q(q^2+1)}{q^4 - 2q^2 - 3}$, and by means of this value of t,

$n = 2m \times \frac{t+q}{q^2+1} = m \times \frac{2q(q^2+1)}{q^4 - 2q^2 - 3}$, or $\frac{m}{n} = \frac{q^4 - 2q^2 - 3}{2q(q^2+1)}$.

From these conclusions it appears that r and s may be expounded by any two numbers in the ratio of $4q(q^2+1)$ to $q^4 - 2q^2 - 3$, and at the same time m and n must be taken in the ratio of $q^4 - 2q^2 - 3$ to $2q(q^2+1)$ respectively where q may be taken at pleasure.

Example 1. If $q = 2$, then $\frac{r}{s} = \frac{4q(q^2+1)}{q^4 - 2q^2 - 3} = \frac{40}{5} = \frac{8}{1}$, and $\frac{m}{n} = \frac{q^4 - 2q^2 - 3}{2q(q^2+1)} = \frac{5}{20} = \frac{1}{4}$; wherefore we may put $r = 8$, $s = 1$, $m = 1$, and $n = 4$; whence $x = (r^2 + s^2)(m^2 + n^2) = 65 \times 17 = 1105$, $y = (r^2 + s^2)(m^2 - n^2) = 65 \times -15 = 975$, and $z = 4rsmn - (r^2 - s^2)(m^2 - n^2) = 128 + 945 = 1073$ which are the roots of three squares that will answer, and differ materially from those found by the first method of solution.

Example 2. Suppose $q = 4$, then $\frac{r}{s} = \frac{16 \times 17}{221} = \frac{16}{13}$, and $\frac{m}{n} = \frac{221}{8 \times 17} = \frac{13}{8}$, therefore we may take $r = 16$, $s = 13$, $m = 13$, and $n = 8$, then $x = 425 \times 233 = 99025$, $y = 425 \times 105 = 44625$, and $z = 86528 - 9135 = 77393$ being the roots of three other squares that will answer the question.

Example 3. If $q = 5$, then $\frac{r}{s} = \frac{20 \times 26}{572} = \frac{10}{11}$, and $\frac{m}{n} = \frac{572}{10 \times 26} = \frac{11}{5}$, therefore we may take $r = 10$, $s = 11$, $m = 11$, and $n = 5$. Hence $x = 221 \times 146 = 32266$, $y = 221 \times 96 = 21216$

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21216, and $z = 24200 + 2106 = 26216$, which are alſo the roots of three ſquares that will anſwer; or taking their halves, viz. 16133, 10608, and 13108 they will be the roots of three ſquares that will anſwer.

A different value of t may be obtained as follows.

It has been ſhewn that $4t^4+8t^3q+t^2(6q^2-q^4+3)+4tq(q^2+1)$ muſt be a ſquare; wherefore aſſume $2t^2 + 2tq - \frac{q^4 - 2q^2 + 3}{4}$ for its root, and then $4t^4 + 8t^3q + t^2(6q^2-q^4+3)+4tq(q^2+1) = \left(2t^2+2tq - \frac{q^4 - 2q^2 - 3}{4}\right)^2 = 4t^4 + 8t^3q + t^2(6q^2-q^4+3) - tq(q^4 - 2q^2 - 3) + \left(\frac{q^4-2q^2-3}{16}\right)^2$,

which being properly reduced gives $t = \frac{r}{s} = \frac{(q^4 - 2q^2 - 3)^2}{16q(q^2 + 1)^2}$ in which q may be taken at pleaſure.

If $q = 1$, then $t = \frac{r}{s} = \frac{16}{16 \times 4} = \frac{1}{4}$; whence $n = 2m \times \frac{t + q}{q^2 + 1} = \frac{5m}{4}$, or $\frac{m}{n} = \frac{4}{5}$; hence it appears, that we may take $r = 1$, $s = 4$, $m = 4$, and $n = 5$; whence $x = (r^2 + s^2)(m^2 + n^2) = 17 \times 41 = 697$, $y = (r^2 + s^2)(m^2 - n^2) = 17 \times -9 = -153$, and $z = 4rsmn - (r^2 - s^2)(m^2 - n^2) = 320 - 135 = 185$, which are the roots of three ſquares that will anſwer, and are properly the roots of the leaſt numbers that have the required property.

Or the ſolution may be as follows.

Let x^2, y^2, and z^2 denote the required ſquares, then by the queſtion $x^2 - y^2 = a^2$, and $x^2 - z^2 = b^2$, the difference of theſe two equations is $y^2 - z^2 = b^2 - a^2$, which muſt alſo be a ſquare by the queſtion: put $y = b - rv$, and $z = a - sv$, then $y^2 - z^2 = b^2 - a^2 = (b - rv)^2 - (a - sv)^2 = b^2 - a^2 - 2rbv + 2sav + r^2v^2 - s^2v^2$, which gives

$v = \frac{2rb - 2sa}{r^2 - s^2}$; whence $y = b - rv = \frac{2rsa - b(r^2 + s^2)}{r^2 - s^2}$, & $z = a - sv = \frac{a(r^2 + s^2) - 2rsb}{r^2 - s^2}$

Now $y^2 - z^2 = b^2 - a^2$ muſt be a ſquare number by the queſtion, therefore put $(s^2 - r^2)(m^2 - n^2) = b$, and $(s^2 - r^2)\,2mn = a$, then, $y =$

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$y = (r^2+s^2)(m^2+n^2)-4rsmn$, and $z = 2rs(m^2+n^2)-2mn(r^2+s^2)$; from whence and from the firſt equation,

$x^2 = a^2 + y^2 = (s^2-r^2)^2 4m^2n^2 + (r^2+s^2)^2(m^2+n^2) - 8rs(r^2+s^2)mn(m^2+n^2) + 16r^2s^2m^2n^2$

$= (r^2+s^2)^2(m^2+n^2) - 8rs(r^2+s^2)mn(m^2+n^2) + (r^2+s^2)^2 4m^2n^2 =$ a ſquare.

Aſſume $x = 4rsmn - (r^2+s^2)(m^2-n^2)$, then will

$x^2 = (r^2+s^2)^2(m^2+n^2)^2 - 8rs(r^2+s^2)mn(m^2+n^2) + (r^2+s^2)4m^2n^2$

$= [4rsmn - (r^2+s^2)(m^2-n^2)]^2 = 16r^2s^2m^2n^2 - 8rs(r^2+s^2)mn(m^2-n^2) + (r^2+s^2)^2(m^2-n^2)^2$,

which by proper reduction gives $m = n \times \frac{2rs(r^2+s^2)}{r^4+s^4}$; put $n = r^4 + s^4$; then $m = 2rs(r^2+s^2)$, where r and s may be taken at pleaſure.

Example. If $r = 2$ and $s = 1$, then $n = r^4 + s^4 = 17$, and $m = 2rs(r^2+s^2) = 20$; whence $x = 4rsmn - (r^2+s^2)(m^2-n^2) = 2165$, $y = (r^2+s^2)(m^2+n^2) - 4rsmn = 725$, and $z = 2rs(m^2+n^2) - 2mn(r^2+s^2) = -644$.

Prob. II. *By Mr.* James Cunliffe.

To find three ſquare numbers, ſuch, that the ſum of every two of them may be a ſquare number.

Solution.

Let x^2, y^2, and z^2 denote the three ſquares:—then by the queſtion $x^2 + y^2 = a^2$, $x^2 + z^2 = b^2$, and $y^2 + z^2 =$ a ſquare; from the two firſt of theſe equations $x^2 = a^2 - y^2 = b^2 - z^2$. It appears, therefore, from hence, that a, and y, muſt be ſo expreſſed, that the difference of their ſquares $a^2 - y^2$ may be a rational ſquare, whoſe root is the value of x; and moreover, b and z, muſt be determined in ſuch a manner from thence, that $a^2 - y^2 = b^2 - z^2$. From what is done in the laſt ſolution to the preceding queſtion it is obvious that we may take $(s^2-r^2)(m^2+n^2) = a$, and $(s^2-r^2)\,2mn = b$, whence $x = (s^2-r^2)(m^2-n^2)$; for then $z = 2rs(m^2+n^2) - 2mn(r^2+s^2)$. Wherefore

$y^2 + z^2 = (s^2-r^2)^2 4m^2n^2 + 4r^2s^2(m^2+n^2)^2 - 8rs(r^2+s^2)mn(m^2+n^2) + (r^2+s^2)^2 4m^2n^2 = 4r^2s^2(m^2+n^2)^2 - 8rs(r^2+s^2)mn(m^2+n^2) + (r^2+s^2)^2 8m^2n^2 =$ a ſquare by the queſtion. Aſſume $2rs(m^2-n^2) + (r^2+s^2)\,2mn$ for the root of this ſquare; then will

$4r^2s^2(m^2+n^2)^2 - 8rs(r^2+s^2)mn(m^2+n^2) + (r^2+s^2)^2 8m^2n^2 = [2rs(m^2-n^2) + (r^2+s^2)2mn]^2 = 4r^2s^2(m^2-n^2)^2 + 8rs(r^2+s^2)mn(m^2-n^2) + (r^2+s^2)^2 4m^2n^2$, which by reduction gives $n = m \times \frac{4rs}{r^2+s^2}$; put $m = r^2 + s^2$, and then $n = 4rs$, where r and s may be taken at pleaſure.

Example.

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Example. If $r = 1$ and $s = 2$, then $m = r^4 + s^4 = 5$, and $n = 4rs = 8$, whence $x = (s^4 - r^4)(m^4 - n^4) = 3 \times -39 = -117$, $y = (s^4 - r^4)2mn = 3 \times 80 = 240$, and $z = 2rs\,(m^4 + n^4) - (r^4 + s^4)\,2mn = 4 \times 89 - 5 \times 80 = -44$. Therefore 240, 117, and 44 are the roots of three squares the sum of every two of which will be a square, and are the least that have hitherto been found. The discovery of these numbers is attributed to EULER; but whoever was the first discoverer, they are obviously the first numbers that present themselves by the above method of solution.

(To be continued.)

ARTICLE III.

ON CONTINUED SURDS, &c.

By Mr. BENJ. GOMPERTZ.

I. IF $x^2 - x = a$, x will be equal

$$\sqrt{a + x} = \sqrt{a + \sqrt{a + x}} = \sqrt{a + \sqrt{a + \sqrt{a + x}}} = \sqrt{a + \sqrt{a + \sqrt{a}}}, \text{ adinfinitum.}$$

II. If $x = \sqrt[3]{a^2 b}$, x^3 will be $= a^2 bx$, and therefore,

$$x = \sqrt{a + \sqrt{bx}} = \sqrt{a\sqrt{b\sqrt{a\sqrt{bx}}}} = \sqrt{a\sqrt{b\sqrt{a\sqrt{b}}}}, \text{ adinfinitum.}$$

Which demonstrates *John Bernoulli's* Solutions, See *Emerson's Miscellanies, page* 433; these, Emerson affirms to be expressions of infinite value, and on that account that Bernoulli's Solutions are wrong. That they are not of infinite value is plain from above, as also from the solutions in the Miscellanies; and that the expressions are not even diverging may be thus generally demonstrated.

1. If a be equal or greater than unity, $\sqrt{a + \sqrt{a}}$ may be proved by the binomial theorem to be less than $\sqrt{a} + \frac{1}{2}$, therefore $\sqrt{a + \sqrt{a + \sqrt{a}}}$ is less than $\sqrt{a + \frac{1}{2} + \sqrt{a}}$, and therefore

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therefore by all means less than $\sqrt{a+\frac{1}{2}+\sqrt{a+\frac{1}{8}}}$, or its equal $\sqrt{c+\sqrt{c}}$, (putting $c=a+\frac{1}{2}$), and still farther less than $\sqrt{c}+\frac{1}{2}$, therefore $\sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a}}}}$ is less than $\sqrt{a+\frac{1}{2}+\sqrt{c}}$, or its equal $\sqrt{c+\sqrt{c}}$, therefore far less than $\sqrt{c}+\frac{1}{2}$, and proceeding thus we continually find the same expression $\sqrt{c}+\frac{1}{2}$ to be greater than $\sqrt{a+\sqrt{a+\sqrt{a+}}}$ &c. and so much the more so as we proceed.

Or thus whether a be greater or less than unity.

Put $b=a+1$, therefore $\sqrt{a+\sqrt{a}}$ is less than $\sqrt{b+\sqrt{b}}$ and still less than $\sqrt{b}+\frac{1}{2}$, (b being greater than unity); therefore $\sqrt{a+\sqrt{a+\sqrt{a}}}$ is less than $\sqrt{a+\frac{1}{2}+\sqrt{b}}$ and still less than $\sqrt{b+\sqrt{b}}$ and still less than $\sqrt{b}+\frac{1}{2}$; therefore $\sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a}}}}$ is less than $\sqrt{a+\frac{1}{2}+\sqrt{b}}$ and still less than $\sqrt{b+\sqrt{b}}$ and still less than $\sqrt{b}+\frac{1}{2}$, and thus we continually get the same expression $\sqrt{b}+\frac{1}{2}$ to be greater than $\sqrt{a+\sqrt{a+\sqrt{a+}}}$ &c. and the more so the further we proceed.

That the expression is greater than a needs no demonstration.

2. Here the expression may be written thus

$$a^{\frac{1}{2}}b^{\frac{1}{4}}a^{\frac{1}{8}}b^{\frac{1}{16}}a^{\frac{1}{32}}b^{\frac{1}{64}}\text{ &c.} =$$

$$a^{\frac{1}{2}+\frac{1}{8}+\frac{1}{32}+}\text{ &c.} \times b^{\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+}\text{ &c.}$$

$=$ (because the converging series $\frac{1}{2}+\frac{1}{8}+\frac{1}{32}+$&c. $=\frac{2}{3}$ and the converging series $\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+$ &c. $=\frac{1}{3}$) $a^{\frac{2}{3}}b^{\frac{1}{3}}=\sqrt[3]{a^2b}$ the same as above, and it is here evident that the series converge.

III. If $x^3-2ax=b$, we have $x^3-2ax=bx$, and therefore

$$x=\sqrt{+a\sqrt{bx+a^2}}=\sqrt{a+\sqrt{a^2+b\sqrt{a+\sqrt{bx+a^2}}}}$$

$$=\sqrt{a+\sqrt{a^2+b\sqrt{a+\sqrt{a^2+b\sqrt{a+\sqrt{bx+a^2}}}}}}$$

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$$= \sqrt{a + \sqrt{a^2 + b\sqrt{a + \sqrt{a^2 + b\sqrt{a + \sqrt{a^2}}}}}}, + \textit{adinfinitum}.$$

IV. Let $x^3 + ax = b$; therefore $x = \frac{1}{a}b + \frac{1}{a}x^3$

$$= \frac{1}{a}b - \frac{1}{a} \times (\frac{1}{a}b - \frac{1}{a}x^3)^3 = \frac{1}{a}b - \frac{1}{a} \times \left\{\frac{1}{a}b - \frac{1}{a} \times (\frac{1}{a}b - \frac{1}{a}x^3)^3\right\}^3$$

$$= \frac{1}{a}b - \frac{1}{a} \times \left\{\frac{1}{a}b - \frac{1}{a} \times [\frac{1}{a}b - \frac{1}{a} \times (\frac{1}{a}b, \textit{adinfinitum})^3]^3\right\}^3, \text{ and putting}$$

$c = \frac{b}{a}$ it becomes $c - \frac{1}{a} \times \left\{c - \frac{1}{a} \times [c - \frac{1}{a} \times (c, \textit{adinfinitum})^3]^3\right\}^3$.

The computation by these series will be plain, by the following EXAMPLE. Let $x^3 - 10x = -3$, to find x. Compare this with the last theorem, and we have $a = -10, b = -3, c = .3$, $c^3 = .027$, $c - \frac{1}{a} \cdot c^3 = .3027$ nearly $= x$, or $c - \frac{1}{a} \cdot (c - \frac{1}{a} \cdot c^3)^3 = .3 + .002773 = .302773$, which differs from the true value of x but by 2 in the last place, and had we proceeded we might have had any degree of accuracy. Though what I have done here may not be of great importance, it may still be acceptable as it confirms Bernoulli's solutions.

ARTICLE IV.

Answers to the Mathematical Questions proposed in ARTICLE XLIV. Vol. II.

I. QUESTION 211, *answered by Mr.* Gregory, *of Cambridge.*

WHEN it is required to determine the sun's altitude at 6 o'clock, and his amplitude at rising, from the latitude and declination given, the two following analogies are generally made use of:

1. As radius : sine latitude : : sine of sun's dec. : sine alt. at 6.
2. As cos. latitude : radius : : sine of sun's dec. : sine amplitude.

From the nature of which analogies it is manifest, that both the altitude at 6, and the amplitude will increase, as the latitude of the same name as the declination increases. And as neither the altitude

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altitude nor amplitude increaſes very faſt or very irregularly, the former varying from 0° in lat. 0°, to about 15° in lat. 90°; and the latter from 15° 5′ 24″ (= the declination) in lat. 0°, to about 30° in lat. 60°, the method of Trial and Errour, may here be adopted with ſufficient accuracy and ſucceſs.

Now, it is obvious upon ſlight conſideration, that the latitude ſought muſt be between 45° and 55°: ſuppoſe therefore, that the latitude is 50°; then, applying the above analogies, we get 11°. 30′. 13″ for the altitude at 6, and 23°. 53′. 30″ for the amplitude: the ſum of theſe and the declination is 50°. 29′. 7″ inſtead of 50°; the errour, therefore, is 29′. 7″ or 1747″ in exceſs. Again, ſuppoſe 52° the latitude; then, we obtain, 11°. 50′. 17″ for the altitude, and 25°. 0′. 54″ for the amplitude: the ſum of theſe and the declination is 51°. 56′. 35″, inſtead of 52°; hence the errour ariſing from this aſſumption is 3′. 25″ or 205″ in defect. Whence by the common rule we find the north latitude = 51·78996 or 51°. 47′. 24″ nearly. The altitude at 6, correſponding to this latitude, and declination 15°. 5′. 24″ is 11°. 48′. 13″½, and the amplitude 24°. 53′. 25″½; the ſum of theſe and the declination differ not more than 1′ from the latitude; thus confirming the truth of the reſult. In this ſolution no notice has been taken of the change in the ſun's declination, between the ſun's riſing and noon of the given day.

The ſame, anſwered by the Propoſer, Mr. Croudace.

Put the ſine and coſine of the declination = a and b, and the ſine of the latitude = x. Then, by Spherics, the ſine of the altitude at 6, is = ax, and the ſine of the amplitude = $a \div \sqrt{(1-x^2)}$. But, by *Emerſon's Trigonometry, page* 21, ſine alt. × cos. ampl. + cos. alt. × ſine ampl. is = to ſine lat. × cos. decl. — cos. lat. × ſine decl. that is,

$$ax \times \sqrt{\frac{c-x^2}{1-x^2}} + \sqrt{(1-a^2x^2)} \times \sqrt{\frac{a^2}{1-x^2}} = bx - a\sqrt{(1-x^2)},$$ by

putting $c = 1 - a^2$. This equation reduced gives,
$(bx \div a) \times \sqrt{(1-x^2)} - \sqrt{(cx^2 - x)} - \sqrt{(1-a^2x^2)} + x^2 = 1$.
Hence $x = ·784429$ anſwering to 51°. 40′. 5″ north latitude.
Anſwers were alſo received from Meſſrs. Barron, Buſſham, Cunliffe, Davis, Lowry, May, Morones Minor, *and* Thornoby.

II. QUESTION 212, *anſwered by Miſs* Suſan May.

It is evident that the objects A, B, C, D are in the circumference of a ſmall circle of the ſphere, whoſe diameter may be

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determined from the diſtances given in the queſtion. Now theſe diſtances are portions of a great circle of the ſphere, and conſequently the chords ſubtended by theſe diſtances may be found by reverſing ſome of the rules in *Dr. Hutton's Menſuration*, and the diameter of the circle may be found by *Prob.* 98, *of Emerſon's Algebra*.

But in the preſent caſe where the diſtances are ſo ſmall, that the difference between the arcs and chords are inconſiderable, we ſhall avoid the trouble of calculating the lengths of the chords, by uſing the arcs in their ſtead.....Now, by the Prob. above-mentioned, we obtain 33·1784 for the diameter of the circle, and if O (fig. 497, pl. 27.) be the centre of the earth, GF the height of the hill, FE a tangent to the earth at E and EH be $\perp$ to OF, we have, (by the ſim. $\triangle$s OFE, OEH,) $OE^2 = GH \times HF$: but OE is $= 3978{\cdot}87$, and $EH = 16{\cdot}5892 = \frac{1}{2}$ the diameter above found, therefore $OF = OE^2 \div OH = OE^2 \div \sqrt{(OE^2 - EH^2)} = 3978{\cdot}904$; conſequently $GF = {\cdot}034$ miles $= {\cdot}034 \times 5280 = 179{\cdot}5$ feet nearly the height required.

The ſame, anſwered by Merones Minor.

Here are given the four ſides of a trapezium inſcribed in a circle to find the diameter, which, *by Simpſon's Select Exercises, Prob.* 35, is $= 33{\cdot}1783$ miles. Then $25000 : 33{\cdot}1783 \div 2 :: 360^\circ : 14'.\ 20''$, the $\angle$ at the centre of the earth ſubtended by the radius of the above circle; the natural ſecant thereof *minus* radius $= {\cdot}0000087$ miles for the height of the hill to radius 1, which, $\times (25000 \times 5280) \div (2 \times 3{\cdot}14159$ &c.) gives $182{\cdot}8$ feet, the height required, by Gardner's Logarithms. Or by uſing Hutton's Logarithms the height $= {\cdot}0000085$ miles (to radius 1) $= 178{\cdot}6$ feet.

This queſtion was alſo anſwered by Meſſrs. Barron, Buffham, Cunliffe, Davis, Gregory, Lowry, *and* Thornoby.

III. QUESTION 213, *anſwered by Miſs* Suſan May.

Fig. 498, Pl. 27. Let ACE, ADF be the ſegments, draw the diameter AQI and join AC. Then, *by Rule V. of Dr. Hutton's Menſuration*, $(2CB + \frac{4}{3}CA) \times \frac{4}{10}AB =$ the area of the ſegment ACE, and $\cdot 7854 \times AB^2$ is $=$ the area of the greateſt inſcribed circle; therefore by the Queſtion
$(2CB + \frac{4}{3}CA) \times \frac{4}{10}AB : {\cdot}7854 \times AB^2 :: 5 : 2$, or
$CB + \frac{2}{3}CA : AB :: 1 : {\cdot}407403$, or
$CB + \frac{2}{3}CA : \sqrt{(CA^2 - CB^2)} :: 1 : {\cdot}407403$. But (*Rule VI. ibid.*) $\frac{4}{3}CA - CB = \frac{5}{2} \times$ Arc $CAF = 75$, wherefore
$CA = \frac{75}{4} + \frac{1}{4}CB$, and $CA^2 = (\frac{75}{4})^2 + \frac{150}{4} \times CB + \frac{1}{16} \times CB^2$. Theſe val-

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values being ſubſtituted in the above proportion it becomes
$\frac{7.5}{6}+\frac{7}{6}\times CB : \sqrt{\left(\frac{75^2}{16}+\frac{150}{4}\times CB-\frac{15}{16}\times CB^2\right)} :: 1 :: .407403$.
Hence by means of a quadratic equation we obtain CB = 18·8 nearly, CA = 23·45, and BA = 14·025; conſequently $AC^2 \div AB = AQ = 39·2$, the diameter of the leaſt circle.

Again, *by Rule V. ibid.*
$2AI \times \sqrt{\left\{3AB \div (3AI - AB)\right\}}$ = the arc DAF = 60, or $4AI^2 \times 3AB \div (3AI - AB) = 60^2$. Therefore $AI^2 = (\frac{60}{2})^2 \times AI - 60 \times 5$; conſequently AI = 59·075 the diameter of the greater circle. Wherefore $(AI^2 - AQ^2) \times .7854 = 1534$ ſquare yards, the area included between the arcs of the circles, nearly.

The same, anſwered by Mr. Geo. Buffham, *Boſton.*

Put $50 = l$, $.7854 = p$, AC = y, and CB = x. Then by the common rule for finding the length of an arc, $8y - 2x = 3l$, or $y = \frac{1}{4}x + \frac{3}{8}l$; whence $AB = \sqrt{(y^2 - x^2)} = \sqrt{(\frac{1}{16}lx - \frac{15}{16}x^2 + \frac{9}{64}l^2)}$, and the area of the circle AGBK $= p \times (\frac{1}{16}lx - \frac{15}{16}x^2 + \frac{9}{64}l^2)$. By Dr. Hutton's Menſuration, 4to edit. Rule IV. for finding the area of a circular ſegment, the area CAE $= (\frac{1}{15}x + \frac{1}{5}l) \times \sqrt{\frac{3}{16}lx - \frac{15}{16}x^2 + \frac{9}{64}l^2}$. By the Queſtion $5 : 2 :: (\frac{1}{15}x + \frac{1}{5}l) \times \sqrt{(\frac{3}{16}lx - \frac{15}{16}x^2 + \frac{9}{64}l^2)} : p \times (\frac{3}{16}lx - \frac{15}{16}x^2 + \frac{9}{64}l^2)$; or $5p \times (\frac{3}{16}lx - \frac{15}{16}x^2 + \frac{9}{64}l^2) = (\frac{2}{15}x + \frac{2}{5}l) \times \sqrt{(\frac{3}{16}lx - \frac{15}{16}x^2 + \frac{9}{64}l^2)}$. This by reduction gives $x^2 - 3·8963\dot{6}x = 279·8783$, or $x = 18·79$. Hence the diameter of the leſs circle is = 39·2. And from the ſame rule the diameter of the greater circle is = 59·0707, and the difference required = $1533\frac{2}{3}$ ſquare yards nearly.

Other ſolutions were received from Meſſrs. Barron, Davis, Gregory, Hill, Lowry, *and* Swale.

IV. QUESTION 214, *anſwered by Mr.* John Barron, *Spilſby.*

Fig, 499, Pl. 27. Put D = AB = 6 inches the diameter of the baſe, b = the altitude = 4 inches, m = a third part of the ſolidity = 12·5664, and FB = x. Then, $D : b :: x : bx \div D = DI$ the altitude of the part cut off. And by *Rule VI, Dr. Hutton's Menſuration*, the area of the part HBG is $= \frac{4}{3}x \times \sqrt{(Dx - \frac{3}{5}x^2)}$; therefore *by Rule to Prob.* 28, *page* 232, *ibid*, we have
$[\frac{1}{3}D \times \sqrt{(Dx - \frac{3}{5}x^2)} - \frac{4}{5}(D - x) \times \sqrt{(Dx - x^2)}] \times (bx \times D) = m$.

 Hence

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Hence by the method of trial and errour, x is found $= 3^{\circ}21$ nearly for the firſt man's ſhare. Again, by taking $m = \frac{2}{3}$ of the ſolidity $= 25^{\circ}1327$, we find $x = 4^{\circ}4$ nearly for the ſecond man's ſhare.

Meſſrs. Buffham, Davis, Gregory, Hill, Lowry, *and* May, *alſo ſent ſolutions to this queſtion.*

V. QUESTION 215, *anſwered by Miſs* Suſan May.

In Fig. 500, Pl. 27. Let C be the centre of the earth's ſhadow, O the centre of the moon at the firſt obſervation, A the centre at the middle of the eclipſe, or when the obſcuration is the greateſt, I that at the beginning, and H that at the end.—— About the centre C with the diſtance CO deſcribe a circle DOBP for the boundary of the earth's ſhadow, and let CAP be drawn, which will be $\perp$ to HI, the relative path of the moon's centre. Join BC, BD, BO, PO, CO, CD, CH and CI. Then by the queſtion, BD the diſtance between the illuminated cuſps is given $= 30'.15''$, therefore its half, BE, is given $= 15'.7''.5 = 907''.5$, and the ſemidiameter BO is given $= 15'.24'' = 924''$; therefore $OE = \sqrt{(BO^2 - BE^2)} = 173^{\circ}83''$, and, *by Lanſon's 43d Prop.* (vide Repoſ. No. X.) $OC = CP = BO^2 \div 2OE = 2455''.65$.

Again as, $6^{\circ} : 15'.24'' :: 1^{\circ}.27'.36'' : 224''.84 = AP$; therefore $AC = CP - AP = 2230''.81$ and $AO = \sqrt{(OC^2 - AC^2)} = 1026''.52$. Moreover $CI = CO + OB = 3379''.65$,

and therefore $AI = \sqrt{(CI^2 - CA^2)} = 2535''.89$. Now as $29'.45''$ (the horary motion) : $60' :: \begin{cases} AI : 1h\ 25m.\ 14s. \\ AO : oh.\ 34m.\ 30s. \end{cases}$ Hence

11h. 49m. 48s. + 34m. 30s. = 12h. 24m. 18s, the middle,
12h. 24m. 18s. — 1h. 25m. 14s = 10h. 59m. 4s, the beginning, and
12h. 24m. 18s. + 1h. 25m. 14s = 13h. 49m. 32s, the ending.

The ſame, anſwered by Merones Minor.

The figure being drawn as before, biſect OD by the $\perp$ Ch. Then in the $\triangle$ DEO we have DO, DE to find the $\angle$ DOE $(CO\hat{h}) = 79^{\circ}.9'.35''$, and in the $\triangle$ $CO\hat{h}$ we have all the $\angle$s and the ſide Oh to find $CO = 2455''.64$. Then ſay as $6^{\circ} : 1^{\circ}.27'.36'' :: 15'.24'' : 224''.84$, and $2455''.64 - 224''.84 = 2230''.8 = AC$; alſo $CO + DO = CI = 3379''.64$. Now in the two right $\angle$d $\triangle$s CAO, CAI we have given CA, CO, CI to find $AO = 1026''.5$, and $AI = 2235''.9$, which, by the given horary motion of the moon from the ſun will be paſſed over in

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in 34'. 30''. 25 and 1h. 25'. 14''·37 respectively. Hence by addition and subtraction,

the beginning was at 10h. 59m. 3·88s.
the middle was at 12h. 24m. 18·25s.
and ending was at 13h. 49m. 32·62s.

*** In these solutions the moon's relative orbit has been considered as a straight line, and her motion therein as equable; since it was not the wish of the proposer to render the solution of the question more intricate by considering them otherwise.

Ingenious solutions to this question were received from Messrs. Davis, Gregory, Hill *and* Lowry.

VI. QUESTION 216, *answered by Mr.* John Andrew, *of Cark.*

CONSTRUCTION. Fig. 501, Pl. 27. With the given diameter DO describe a circle, and ⊥ thereto inscribe AB = the given base; join OB, to which add, *by the* 18, *V. of Simpson's Geometry*, the line B*g*, so that O*g* × *g*B may be = to the rectangle under DO and the given difference between the ⊥ and rad. of the inscribed circle. From O, to the circle, apply OC = O*g*; join AC, CB and ACB will be the △ required.

DEMONSTRATION. Join DC, CO, and drop the ⊥ CP; make OQ = OB, and draw QG ∥ AB. Now, it is well known, that Q is the centre and GP the radius of the circle inscribed in the △ ACB. Moreover by similar triangles

OD : OC = O*g* : : CQ = B*g* : CG, hence
O*g* × *g*B = OD × CG, and, by the construction,
O*g* × *g*B = OD × the given difference;
therefore CG is equal to the given difference. *Q. E. D.*

The same answered by Mr. James Cunliffe, *of Bolton.*

ANALYSIS. Let ACB be the required △ inscribed in the given circle, DO a diameter of the circle bisecting the base AB in E; join OB and OC.

Upon OC take OQ = OB, and Q is the centre of the inscribed circle, as is very well known. Draw QG ∥ to AB meeting the ⊥ CP in G, and join DC. The △s ODC, and QCG are right angled and similar, therefore

OD : OC = OB + CQ : : CQ : CG; whence by Eu. 16. VI.
CQ × (CQ + OB) = OD × CG; now OD, OB and CG are all given, and CQ may be found *by Prob.* 18. *Book V. Simpson's Geometry*, and from thence OC becomes known.

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The same answered by Mr. W. Peacock, *Birmingham.*

In a circle, whose diameter is the given line DO, inscribe the chord AB, $\perp$ to DO and equal to the given base. With the centre O and distance OA, or OB, describe the circle AIQB cutting DO at I. Make OD : OF :: IF : the given difference (Eu. VI. 29. Playfair's Ed.), and from O, to the circle apply OC = OF, and join AC, BC, so shall ACB be the $\triangle$ required.

For it is well known that the circle AIB is the *locus* of the centre of the inscribed circle, and if CP be drawn $\perp$, and QG $\parallel$ to AB, we shall have by sim. $\triangle$s, OD : OC = OF : QC = FI : CG; therefore it is manifest that CG is = to the given difference.

The same, also by Mr. James Whalley, *of Bolton.*

Analysis. Suppose DO the diameter of the given circle, AB, $\perp$ to DO, the given base, and ACB the $\triangle$ required. Draw C*d* $\parallel$ to AB, then will *d*E represent the $\perp$ in which take *du* = the given difference. Now (vide No. 3. of the Student) E*n* × O*n* = *dn* × OE, but *dn* and OE are both given, therefore (by *Simpson's Geo.* 18. V.) produce OE to *n* so that O*n* × *n*E may be = to the given rectangle *dn* × OF, then will E*d* be given, and all the other parts of the diameter will from thence be given, and the method of Construction evident enough.

Very neat answers to this question were also received from Messrs. Burdon, *the proposer*, Davis, Gregory, Hill, Lowry, *and* Swale.

VII. QUESTION 217, *answered by the proposer, Mr.* Bosworth.

The best method perhaps of answering Questions of this nature is, first, to find the centre of gravity of the lever, and then, supposing the whole matter it contains to be collected in that point, to find the common centre of gravity of the lever and the globe attached to it, which will be the place of the fulcrum. To apply this method in the present instance: the lever being equally thick and dense throughout, its centre of gravity is in the middle, that is, the length being 10 feet, at the distance of 5 feet from each end. A globe of lead, $4\frac{1}{2}$ inches diameter, weighs 17lbs, and as the weights of similar solids are in proportion to the cubes of their like dimensions we have $(4\frac{1}{2})^3 : 17 :: 6^3$, (the cube of the diam. of the given globe) : 47·8339lbs. weight of the given globe. Now, as the distance of the common centre of gravity of these bodies, and consequently of the fulcrum, from each end, is inversely as the weights, we have as 40 (weight of the lever) + 47·8339

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47·8339 (weight of the globe) : 5 feet : : 40 : 2·27702515 feet, diſtance of the fulcrum from the end at which the globe hangs.

The ſame, anſwered by Miſs Sufan May.

Fig. 502, Pl. 27. Let ADHE be a ſection of the lever, and let G be the point required. Take FG = GH, and draw FB ‖ to HD. Then the equal parts GCDH, GCBF on the different ſides of the fulcrum G, will keep each other in equilibrio; therefore the weight ſuſpended at H muſt be a counterpoiſe for the remaining part of the lever, that is for the part ABEF.

Let *m* be the centre of gravity of the part ABEF, then, by mechanics, the force of the part ABEF, to turn the lever about the fulcrum G, will be the ſame as a weight equal thereto placed at *m*; conſequently as the weight of the part ABEF is to the diſtance *cd* or GH, ſo is the weight of the lead to the diſtance *mc*.—Now ſince *am* = *mb*, and *bc* = *cd*, *mc* is = half the length of the lever 5 feet, and the weight of the part ABEF is = 8*mb*, therefore, if *w* be = to the weight of the lead (= 46·326 lbs.) we have $8mb : 5 - mb :: w : 5$, and, by reduction, $mb = \frac{5w}{40 + w} = 2{\cdot}68321$, conſequently $bc = 5 - mb = 2{\cdot}31679$ the diſtance required.

The ſame, anſwered by Mr. John Barron, *of Spilſby.*

The weight of a leaden ball is = $\frac{2}{9}$ of the cube of its diameter, therefore $6^3 \times \frac{2}{9} = 48$ lbs. the weight of the given ball.

Let *ad* be = 10 the len. of the lever, *cd* = *x*, the diſtance from that end of the lever, from whence the ball is hung, to the fulcrum; then $10 - x = ac$ the other part of the lever. Since the weight of every foot of the lever is evidently = 4lb, $4x$ will be the weight of the part GD, and $40 - 4x$ the weight of the part AG. The diſtance of the centre of gravity of the part DG from the point *c* is = $\frac{x}{2}$, and that of the part AG from the ſame point *c* is $\frac{10 - x}{2}$. Hence, by the nature of the queſtion $\frac{x}{2} \times 4x + 48x = \frac{10 - x}{2} \times (40 - 4x)$. From this equation $x = \frac{200}{98} = 2{\cdot}2727$ feet.

The ſame, anſwered by Mr. George Buffham.

Let *ad* repreſent the lever, *w* the globe of lead, the weight of which is found to be 46·326 lbs, and let *c* be the point where the

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the fulcrum is to be placed. Put $cd = y$, then $dc = 10 - y$, the weights of which will be $4y$ and $40 - 4y$. By the property of the lever $(40 - 4y) \times (5 - \frac{1}{2}y) = 4y \times \frac{1}{2}y + 46{\cdot}3267$; whence $y = 2{\cdot}3168$ the diſtance from the end d to the point required.

The ſame, anſwered by Mr. David Davis.

Let F (fig. 503, pl. 27.) be the fulcrum on which the lever AB and weight W are ſuppoſed to reſt *in equilibrio*: then, ſince the lever is equally thick throughout, if FC be equal to FA, the *momenta* of theſe two portions of the lever will be equal the one to the other, and we have to determine when the momentum of the part CB is equal to that of the weight W, both conſidered with reſpect to the fulcrum F. To perform this generally, put x for FB, y for AF or FC, s for 4 lbs, the weight of 1 foot in length of the lever, and w for the weight ſuſpended from A: then yw will denote the momentum of W; and CB being equal to $x - y$, the diſtance FG of its centre of gravity from F being $y + \frac{x-y}{2} = \frac{x+y}{2}$; we have $(x-y) \times s \times \frac{x+y}{2} = \frac{sx^2 - sy^2}{2}$, for the momentum of CB. Hence this equation, $yw = \frac{sx^2 - sy^2}{2}$; which reduced gives $x = \sqrt{(y^2 + \frac{2wy}{s})}$. And from this general equation, any three of the four quantities x, y, w, or s, being given, the fourth may readily be found.

In the caſe before us, where AB $= 10$, we have $x = 10 - y$; which value of x equated with the foregoing, gives $10 - y = \sqrt{(y^2 + \frac{2wy}{s})}$; whence by proper reduction we ſoon obtain $y = \frac{100s}{20s + 2w} = \frac{200}{40 + w}$, s being equal to 4.....Now, it is known, by repeated experiments, that a ball of lead of $4\frac{1}{4}$ inches diameter weighs 17 lbs; wherefore, we have, by the nature of ſimilar ſolids, $(4\frac{1}{4})^3 : 6^3 :: 17$ lbs: $47{\cdot}84391$ lbs, the weight of the given globe, or the value of w; which being ſubſtituted for it in the final equation above, gives $y = 2{\cdot}277024898$ feet, $=$ AF, the diſtance of the fulcrum from the end A.

This queſtion was likewiſe very ingeniouſly anſwered by Meſſrs. Cunliffe, Gregory, Hill, Lowry, Marrat, Merones Minor, Peacock, *and* Swale.

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VIII. QUESTION 218, *answered by Mr.* Cunliffe.

CONSTRUCTION. Fig. 504, Pl. 27. Make a right angled △ BGO′ such, that the leg BG shall be to the hypothenuse BO′ in the given ratio of the base to the diameter of the circumscribing circle: at B take BL ⊥ to BG and such that BG may be to BL in the given ratio of half the difference of the segments of the base to the perpendicular, and draw GL. Then with the centre O′ and radius O′B describe a circle cutting BG produced in A and the line GL in C; join AC, BC, and the △ ACB shall be similar to that required.

DEMONSTRATION. Upon AB drop the ⊥ CP; then because of the parallels PC, BL; GB : BL :: GP : PC, the given ratio of half the difference of the segments of the base to the perpendicular, by the construction; and the rest is evident. Therefore the △ may be constructed either with the radius of the inscribed circle, or any other datum.

The same, answered by Merones Minor.

ANALYSIS. Suppose ACB the required △, DGI, ⊥ to AB, the diameter of the circumscribing circle, and O the centre of the inscribed circle. Draw CP ⊥ to AB and join CG. Then, the ratio of AB to DI being given, the vertical ∠ ACB, is given; and OM, the radius of the inscribed circle, being given, CO is given. Hence this

CONSTRUCTION. Upon any assumed line *di* describe a circle and inscribe *ab* ⊥ thereto, so that *ab* : *di* = AB : DI; take *ep* : *eg* = GP : CP, of any magnitude at pleasure; compleat the parallelogram *e*C*g*, and draw the diagonal *ec*C to cut the circle in C; join C*a*, C*b*, C*i*; in C*i* take CO of the given length, as found in the analysis, and describe the circle HMN, with the given radius of the inscribed circle; finally draw AB ∥ to *ab*, to touch the circle HMN, and cutting C*a*, C*b*, in A and B; so shall ACB be the △ required.

The same, answered by Mr. W. Peacock.

On any line DI as diameter describe a circle and apply AB ⊥ to the diameter and of such a length that its ratio to DI may be the same as the given ratio of the base to the diameter. Let G be their point of intersection, draw DE ⊥ DG, and take 2DE to DG in the other given ratio. Draw GE to meet the circle at C, join AC, BC; the △ ACB is evidently similar to the required one.

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The ſame, anſwered by Mr. James Whalley.

ANALYSIS: Suppoſe the required △ ACB to be circumſcribed by the circle ADBI, whoſe diameter DI is ⊥ to AB the baſe of the △. Let O be the centre of the inſcribed circle; upon AB demit the ⊥s CP, OH, and draw CG, G being the middle of the baſe, draw alſo OA, OB. Now the vertical ∠ ACB is given, becauſe the ratio of AB : DI is given; and, the ratio of CP : PG being given, the △ CPG is given in ſpecies, hence the ∠ CGB is given; therefore the △ ABC is given in ſpecies; wherefore the △ OAB is given in ſpecies; but OH is given in magnitude, therefore the △ OAB is given in magnitude, conſequently AB is given, and a conſtruction is evident.

Neat ſolutions were given to this queſtion by Meſſrs. Barr, Davis, Gregory, Hill, Lowry, May, *and* Swale.

IX. QUESTION 219, *anſwered by Miſs* Sufan May.

Fig. 505, Pl. 27. Let EE′ be the equatorial diameter, NS the polar diameter, C the centre of the earth, and P the place where the apparent latitude is taken. Join PC and draw PB ⊥ to the horizon at P; then (if I rightly comprehend the propoſer's meaning) there is given the ∠ EBP, the meaſure of the apparent latitude, to find a ſimple and accurate expreſſion for the ∠ ECP, the latitude at the centre.

On EC demit the ⊥ PA, and draw BF ∥ to PC. Then by Prop. XV. *Hutton on the ellipſe*, E′E^{a} : NS^{a} :: AC : AB :: AP : AF :: tang. ∠ PBA : tang. ∠ FBA, or its equal the ∠ PCA. Wherefore as the ſquare of the equatorial diameter is to the ſquare of the polar diameter, ſo is the tangent of the apparent latitude to the tangent of the latitude at the centre.

The ſame, anſwered by Mr. T. S. Evans, *the propoſer.*

Let EPN be a quarter of the meridian: EC the equatorial radius which put $= a$; NC the ſemi-axis $= b$. Let PB $= n$ be drawn ⊥ to PT a tangent to P, the place whoſe apparent latitude is given, the ſine of which put $= s$; CA $= x$, and AP $= y$; then by the property of the ellipſis $y^{a} = \frac{b^{a}}{a^{a}}(a^{a} - x^{a})$, likewiſe

$a^{a} - x^{a}$

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$\frac{a^2 - x^2}{x} = AT$; and $AF^2 : AP^2 :: AP^2 : AB^2$ or $(\frac{a^2 - x^2}{x})^2 :$ $\frac{b^2}{a^2}(a^2 - x^2) :: \frac{b^2}{a^2}(a^2 - x^2) : \frac{b^4 x^2}{a^4} = AB^2$; now $x^2 = a^2 - \frac{a^2}{b^2} y^2$, therefore substituting this for x we get $AB^2 = \frac{b^4}{a^2} - \frac{b^2}{a^2} y^2$: but $AB^2 + AP^2 = BP^2 = \frac{b^4}{a^2} - \frac{b^2}{a^2} y^2 + y^2 = n^2$, and $y = sn$, therefore $n^2 = \frac{b^4}{a^2} - \frac{b^2}{a^2} s^2 n^2 + s^2 n^2$, whence

$n = \sqrt{\frac{b^4}{a^2 + b^2 s^2 - a^2 s^2}}$, conseq. $y = \sqrt{\frac{b^4 s^2}{a^2 + b^2 s^2 - a^2 s^2}}$ and $x = \sqrt{\frac{a^4 - a^4 s^2}{a^2 + b^2 s^2 - a^2 s^2}}$.

Now in the $\triangle$ APC, if AC be made radius, AP will be the tangent of $PCA = \frac{y}{x} = \sqrt{\frac{b^4 s^2}{a^4 - a^4 s^2}} = \frac{b^2}{a^2} \times \frac{\text{fine lat.}}{\text{cofine lat.}}$; whence the tangent of the angle PCA, or of the latitude reduced to the earth's centre $= \frac{b^2}{a^2} \times$ tangent of the apparent latitude; which is a very simple expression. The reader may now consult *Bezout*'s Navigation at the end.

This solution being general, for any proportions of the earth's diameters, it may not be amiss to notice the present state of equations for reducing the apparent latitude to that at the centre.—By comparing the measurement of *Maupertuis* in lat. 66°. 20'N with that of *Mason* and *Dixon* in lat. 39°. 12'N, we shall have the proportion of the earth's diameters $\frac{140}{141}$; that of *Maupertuis* with *Bouguer* at the equator, we shall obtain $\frac{216}{217}$; and lastly that of *Caille* in lat. 45°N with his in lat. 49°. 23'N we shall have the proportion $\frac{321}{322}$; now

$\left.\begin{matrix}\frac{140}{141} \\ \frac{216}{217} \\ \frac{321}{322}\end{matrix}\right\}$ gives the maximum of reduction of the apparent latitude to that at the centre. $\left\{\begin{matrix}24'.28'' \\ 15.23 \\ 10.42.\end{matrix}\right.$

Hence

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Hence it appears how uncertain we are reſpecting the preciſe quantity of this equation, and there are other proportions of the earth's diameters which run to a far greater extreme than theſe.—From this alſo appears the great utility of the preſent Trigonometrical Survey carrying on both in England and France.

As I do not find that *Mayer* has mentioned the proportion he uſed in calculating his Table pa. LXXV; it appears, by an inverſe method of calculation, that he uſed the proportion $\frac{2}{2}\frac{3}{3}\frac{0}{1}$; and *La Lande* uſed the proportion $\frac{2}{2}\frac{2}{3}\frac{9}{6}$; which is the ſame as *Sir Iſaac Newton* found from the meaſurements of *Norwood*, *Picart* and the two *Caſſinis*; ſee the Principia B. 3, P. 19.

Meſſ's Gregory, Hill, Lowry, *and* Thornoby *ſent neat ſolutions to this queſtion.*

X. QUESTION 220, *anſwered by Mr.* John Lowry.

Fig. 506, Pl. 27. Through I draw IL parallel to AC, meeting AB at L, and join LH. Draw HI to meet AC at G, and CI to meet AB at M, and join HM. On HL take HO = GC, and draw OQ parallel to IH meeting MH at Q; draw alſo QV parallel to OH meeting CB at V. Then it is manifeſt, *from Prop. IV. of Profeſſor Playfair's Paper on Poriſms*, that if any two lines whatever, as HE, IE be inflected from the points H, I, to the ſtraight line AB, and meet QV in S and AC in F, QS will be equal to FC; and becauſe the angle FCD is given, the triangle FCD will have a given ratio to the rectangle FC·CD, or its equal QS CD; therefore the rectangle QS·CD is of a given magnitude. The problem is therefore reduced to the following, viz. Through the given point H, to draw the ſtraight line DHS ſo, that the ſum or rectangle of QS, CD may be a given magnitude.

The firſt caſe has been done by *Simpſon in his Geometry and Select Exerciſes*, and by *Burrow in his Diary for* 1776, and it may be done rather differently as follows. Draw HX parallel to CV, and HY parallel to QV; then it is evident that the ſum of SX and DY is given; and by ſimilar triangles DY : HY :: HX : SX, or DY·SX = HY·HX = a given rectangle; therefore the ſum and rectangle of DY, SX are given to determine them, the method of doing which is well known.

The rectangle remaining conſtant, the ſum of DY and SX will evidently be the leaſt poſſible when DY is = SX, in which caſe the ſum of QS and CD, or CF and CD will be a maximum.

The ſecond caſe is the ſame as the celebrated problem of the Ancient Geometricians, *De Sectione Spatii*, and may be done as follows. CD = CY — DY and QS = QX — SX; therefore CD·QS = CY·QX — QX·DY — CY·SX + DY·SX: but

DY SX

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DY·SX is given, as is shewn above, it is therefore evident that QX·DY + CY·SX is given; let this be taken = R × QX, and take HW to HY as CY to QX, and draw WU parallel to DY to meet HD (produced if neceſſary) at U. Then, by ſimilar triangles, it is evident that QX·WU is = to CY·SX, therefore DY + WU is = to R, and DY·WU = HX·HW. Wherefore the ſum and rectangle are given to determine each, the ſame as before.

DY + WU will be the leaſt when SX is = to WU, in which caſe the rectangle CD·CF will be a maximum.

If the ratio of CD to CF had been given, the problem would have been reduced to the *Sectio Rationis* of the Ancients. Or, becauſe CY and QV are given lines, DY·SX a given rectangle, and CD : QS a given ratio, CD and QS may be determined by *Prop. III. Caſe 2nd. Geometrical Sections in the Repoſitory*, and this perhaps ſuggeſts as ſimple a ſolution to the celebrated problem of the ancients as any that has yet been given.

The ſame, anſwered by Mr. I. H. Swale.

1. When the ſum of CF, CD is given. Fig. 507, Pl. 27.

CONSTRUCTION. In the ſides CA of the given △ ACB, take CG = the given ſum, join CH, GI, which produce to meet in P; draw HO, PO, parallel to CB, CA meeting in O, make ON : OH in the ratio of the rectangle HC × IP to IG × HP, and join HN; produce HO to AB at T, alſo draw PR parallel to HO meeting AB at R; produce OP to meet AB at S, and draw IV parallel to OP meeting AB in V, and join IR.

Then, from the point T, *by Prob.* 37, *Simp. Geo.* draw the right line TWQU, meeting the given lines IR, HN, IV, in W, Q, U and making WQ : QU = RS : SV. Draw IQ meeting AB, AC, in E, F; draw EH meeting BC in D, join DF, and the △ EDF will be inſcribed as required.

DEMONSTRATION. Let DE, EF, meet RP, SP, in K and L; join KL, RL, and produce RL to meet IV at M.

By Conſtruction,

WQ : QU = RS : SV = RL : LM, per the parallels SL, VM; therefore the lines TQ, RL, are parallel; then,

TR : RE = QL : LE = HK : KE, per the parallels RK, TH; and therefore the lines KL, HQ, are alſo parallel; wherefore

KP : PL = HO : ON, = IG × HP : HC × IP, by the conſt.

Now, becauſe PK is ‖ to CD, and PL ‖ to FG, it follows that IP : HC = PK : CD and IP : IG = PL : FG; therefore

FG × PK : CD × PL = IG × HP : HC × IP = KP : PL, from above.

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Hence TG : CD = PK × PL : PK × PL, or FG = CD.
Therefore CF + CD = CF + FG = CG = the given ſum. *Q. E. D.*

II. When the △ FCD is of a given magnitude. Fig. 508, Pl. 27.

CONSTRUCTION. Demit upon the ſide AC, of the given △ ACB, the perpendicular BR, and make CV² : given magnitude = BC : BR. Join CH, CI, meeting AB at K and G; draw HS, IT, parallel to CB, CA and meeting AB at S and T; and make CU² : CV² = HK × IG : CH × CI. Then, by Wales' Determinate Section, cut ST at E ſo that EK × EG : ES × ET = CU² : SH × TI. Join EH, EI, meeting CB, CA, at D and F; join DF, ſo ſhall the △ CDF be inſcribed as was required.

DEMONSTRATION. Draw KL, GM ‖ to BC, AC, meeting DE, FE in L and M.—By reaſon of the parallels,
EK : ES = KL : SH and EG : ET = GM : TI, therefore
KL × GM : SH × TI = EK × EG : ES × ET = CU² : SH × TI, by conſtr.
therefore KL × GM = CU².—Now,
CH : HK = CD : KL and CI : IG = CF : GM, therefore
CD × CF : KL × GM = CH × CI : HK × IG = CV² : CU² = KL × GM;
therefore CD × CF = CV².—But, by known properties,
CD × CF (CV²) : △ DCF = CB : BR = CV² : given magnitude.
Therefore the △ DCF is = the given magnitude. *Q. E. D.*

Meſſrs. Cunliffe *and* Hill *alſo ſent neat ſolutions to this queſtion.*

XI. QUESTION 221, *anſwered by Miſs.* Suſan May.

Fig. 509, Pl. 27. Let E be the place of the earth, M that of the moon, S that of the ſun, and A the point where their attractions are mutually equal. Then, *by Art.* 382, *Simpſon's Fluxions*, the attraction of ſpherical bodies are directly as the quantity of matter, and inverſely as the ſquare of the diſtance. Conſequently the ratio of the diſtances of the point A from the points E, M, S, will be given. Let EA, MA and SA be drawn; then becauſe E and M are given points, and EA to MA a given ratio, the *locus* of the point A is a circle given in magnitude and poſition, and whoſe centre is in the line EM produced. And, becauſe S is alſo another given point and MA to SA a given ratio, the *locus* of the point A is alſo another circle given in magnitude and poſition, and whoſe centre is in the line SM produced. Conſequently, the interſection of theſe circles will determine the point A. The method of deſcribing theſe circles may be ſeen at page 334 of *Simpſon's Algebra*, and thence the calculation will alſo be evident.

If

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If the circle HAF revolve about its diameter HF, and thereby generate a ſpherical ſuperficies, it is evident that at any point whatever, in that ſuperficies, the attraction of the earth and moon will be equal; and in like manner, if the circle LAR generate a ſpherical ſuperficies by revolving round its diameter LR, any point in the ſuperficies will be equally attracted by the ſun and moon. Now, if we conceive theſe two ſuperficies to interſect each other, the figure of the ſection will be a circle whoſe diameter is AB, the line joining the interſections of the circles HAF, LAR. Conſequently any point whatever in the circumference of that circle is equally attracted by the three bodies.

The ſame, anſwered by Merones Minor.

Let A, B, C repreſent the given magnitudes of the ſun, earth, and moon, reſpectively, and a, b, c, their reſpective diſtances from the point required. Then, the centripetal force being inverſely as the ſquare of the diſtance, which is the GRAND LAW OF NATURE. And, by *Emerſon on Central Forces, Prop. I.* the centripetal force is as the quantity of matter in the attracting body; that is as the given magnitude in the preſent caſe. Therefore $A \div a^2 = B \div b^2 = C \div c^2$, or $\sqrt{A} \div a = \sqrt{B} \div b = \sqrt{C} \div c$, ſo that the queſtion becomes, "To find a fourth point P, from whence, lines drawn to the three given points may obtain a given ratio;" which is *Simpſon's 53rd. Prob. App. Alg.; or the 56th Prob. in his Select Exerciſes.*

XII. QUESTION 222, *anſwered by* Merones Minor.

Put $A = 10$ acres, $= 435600$ feet, $d = 4$, $b = 3$, $g = 16\frac{1}{12}$, $x =$ any variable altitude of the ſurface above the bottom of the apertures, and $t =$ time ſought. Then, by *Dr. Hutton's Select Exerciſes, Prob. 26,* $t = \frac{3A}{4b\sqrt{g}} \times \frac{\sqrt{d} - \sqrt{x}}{\sqrt{dx}}$. But when $x = 0$ this expreſſion is infinite, as there obſerved; if we ſuppoſe $x = \frac{1}{1728}$, then $t = 252083''$, very near, $= 70h.\ 1m.\ 23s.$ the time of exhauſting the pool to within $\cdot 01$ foot of the bottom of the apertures; ſuppoſing the velocity $=$ *that* due to the whole altitude.

The ſame, anſwered by Miſs Suſan May.

This queſtion ſeems deſigned as an example to *Prob. XXVI. of Dr. Hutton's Practical Exerciſes concerning forces,* where it is

 ſhewn

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shewn that if x represent the variable height of the water at any time, b the breadth of the cut (3 feet), d the whole depth (4 feet), A the area of the surface of the water ($= 435600$ feet), and $g = 16\frac{1}{12}$ feet, the general time of sinking the surface to the depth x will be $\frac{3A}{2b\sqrt{g}} \times \frac{\sqrt{d} - \sqrt{x}}{\sqrt{dx}}$. But when x is 0 this expression is infinite, which shews that the water can never be exhausted so as to be on a level with the bottom of the aperture.

By taking $x = \frac{1}{144}$, we obtain, by substituting, the particular values in the above expression, the time of exhausting to $\frac{1}{12}$ of an inch of the bottom $= 624551''$ nearly. Now this is the time of exhausting through one of the apertures: but since an equal quantity is discharged through each, the time will be only one half of that above, that is $= 312275''{\cdot}5 = 86$h. 34m. $35\frac{1}{2}$s.

Mr. GREGORY *of Cambridge* also refers to the same problem in *Dr. Hutton's Select Exercises*, and adopts the general expression $t = \frac{3A}{4b\sqrt{g}} \times \frac{\sqrt{d} - \sqrt{x}}{\sqrt{ax}}$, for the time of sinking the surface to the depth x in both apertures, whence when $x = \frac{1}{1000}$ he deduces the value of $t = 252082{\cdot}88''$. He then observes that when one value of t is thus found, others may be readily ascertained, since they will vary as $\frac{\sqrt{d} - \sqrt{x}}{\sqrt{x}}$. Thus when $x = \frac{1}{1000}$, $\frac{\sqrt{d} - \sqrt{x}}{\sqrt{x}} = 93{\cdot}868328$; but when $x = \frac{1}{100}$, $\frac{\sqrt{d} - \sqrt{x}}{\sqrt{x}} = 19$· : therefore, as 19· : 93·868328 :: 252082·88″ : 1245400·97174″ = 345h. 56m. 40·97s, time of exhausting to the depth of $\frac{1}{1000}$ of a foot.

The same, answered by Mr. John Barron, *Spilsby.*

Put $d =$ the depth of the apertures, $b =$ the breadth of one of them, $g = 16\frac{1}{12}$, $a =$ area of the pool, and $x =$ any variable altitude of the water above the bottoms of the apertures. Then by the Principles of Hydrostatics $2\sqrt{(gx)} =$ the velocity of the effluent

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effluent water, and $2bx\sqrt{(gx)}$ will, therefore, expreſs the fluxion of the quantity of water iſſuing through one of the apertures in a ſecond of time. The fluent of which is $\frac{4bx}{3}\sqrt{(gx)}$, the quantity of water iſſuing through one of the apertures in a ſecond; wherefore, $-a\dot{x} \div \frac{4bx}{3}\sqrt{(gx)} = \frac{3a}{4b\sqrt{g}} \times \frac{-\dot{x}}{x^{\frac{1}{2}}}$ = the fluxion of the time, the fluent of which is $\frac{3a}{2b\sqrt{(gx)}}$. But when $x = d$, the time is nothing, therefore the correct fluent is

$\frac{3a}{2b\sqrt{g}}\left(\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{d}}\right)$ = the time through one of the apertures, and therefore $\frac{3a}{4b\sqrt{g}}\left(\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{d}}\right)$ = the time through both apertures. Now when $x = 0$, this expreſſion for the time becomes infinite, which ſhews that the ſurface of the water in the pool can never arrive at the level of the bottoms of the apertures.

And thus the queſtion was anſwered by Mr. Cunliffe. *It was alſo anſwered by Meſſrs.* Boſworth, Davis, Hill, *and* Lowry.

To XIII. QUESTION 223, *we have received no answer.*

XIV. QUESTION 224, *answered by Mr.* James Cunliffe.

Let ABC (fig. 510, pl. 27.) be the $\triangle$, O the centre of its inſcribed circle, through which and the angular points A and C draw the right lines AE, CD to terminate in the oppoſite ſides BC, AB. Theſe lines, as is well known, will biſect the reſpective angles CAB, and ACB. Let the parts of the weight ſuſtained by the fulcrums at the reſpective angular points be denoted by A, B, and C. It appears from *Emerſon's Mechanics, Prop.* 43, *Sect.* 5, that O is the common centre of gravity of the parts of the weight ſuſtained by each of the fulcrums: moreover it appears from Prop. 45, of the ſame, that D is the centre of gravity of the parts denoted by A and B; and E that of the parts denoted by B and C. Wherefore, ſince CD biſects the $\angle$ ACB, by the nature of the centre of gravity and Euc. 3. VI.

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AC : CB : : AD : DB : : B : A, whence
$A = (BC \div AC) \times B = \frac{4}{5}B$. And for the same reason
$AC : AB : : B : C = (AB \div AC) \times B = \frac{1}{15}B$.
Then by the question $A+B+C = \frac{4}{5}B + B + \frac{1}{15}B = \frac{8}{3}B = 100$.
Whence $B = 37\frac{1}{2}$lb, $A = \frac{4}{5}B = 30$lb, and $C = \frac{1}{15}B = 32\frac{1}{2}$lb.

When each of the props sustain an equal part of the weight, the point O, the place thereof, will be determined by the intersection of the lines CD, AE drawn to bisect AB, BC, in the points D and E respectively.

If O, the place of the weight, be any other given point within the $\triangle$ ABC, the parts of the weight sustained by props at each of the angular points may be determined as follows, *viz.* by drawing through the given point the right lines CD, AE from the $\angle$s C and A to the opposite sides. For it is evident from the above quoted authority that D is the centre of gravity of the parts A and B, and E that of the parts B and C, the ratios of which hence become known.

The same, answered by Mr. Gregory, *the proposer.*

In the solution of this question it is necessary, in the first place, to enquire, when a plane triangle is supported by three fulcra at the the angular points, and a weight W is laid upon it at any other point, what is the ratio of the pressures on each fulcrum? To ascertain this, let ABC (fig. 511, pl. 27.) be the triangle, and D the point where the weight is placed; produce BD to F, and from the points B and D let fall the perpendiculars BH, DG, upon the side AC. Then, if we suppose the triangle to revolve about AC as an axis, the point F may be considered as a fulcrum to the power P at B, and weight W at D: therefore, P : W : : DF : BF : : DG : BH, by similar triangles. Consequently $P = \frac{DG \times W}{BH}$. But W and BH are constant, therefore $P \propto DG$, and of course $P \propto DG \times AC$, or $\propto \frac{DG + AC}{2}$; that is, *the pressure at B is as the area of the triangle DAC opposite.* In the same manner it might be shewn, that the pressures at A and C, are as the areas of the triangles BCD and BAD respectively.

Hence, the first requisite of the question is easily determined: for when the weight is laid at D the centre of the inscribed circle, the perpendiculars DF, DI, and DL, let fall on the sides, will be equal to each other, and the areas of the triangles ACD, ABD, BCD, will be as their bases AC, AB, BC; therefore, by what has been shewn above, the pressure upon the angles A, B, C, will be as BC, AC, and AB, or as 26, 30, and 24 respectively.

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It

It is alſo manifeſt from the foregoing, that, when the preſſure is the ſame on each fulcrum, the point D muſt be ſo choſen that lines drawn from it to the angular points ſhall divide the triangle into three equal parts. In this caſe the point D may be found by the following

CONSTRUCTION: Biſect AB in the point E by the line CE, make AP = $\frac{1}{3}$AC, and draw PK parallel to AB: the point D in which CE and PK interſect each other, is the point required. ....For, ſince CP = $\frac{2}{3}$AC, and the triangles CPD, CAE are ſimilar; ſimilar triangles being as the ſquares of their homologous ſides, we have CPD = $\frac{4}{9}$CAE, alſo, ſince AP = $\frac{2}{3}$PC, and the triangles APD, CPD, are between the ſame parallels, APD = $\frac{2}{3}$CPD = $\frac{2}{9}$CAE: therefore ACD, the ſum of theſe, = $\frac{6}{9}$($\frac{1}{3}$) CAE, or $\frac{1}{3}$ABC, the whole triangle being biſected by the line CE. In like manner, ſince CK = $\frac{2}{3}$BC, it might be ſhewn that the triangle CDB = $\frac{1}{3}$ABC; and conſequently ADB, the remaining triangle, = $\frac{1}{3}$ABC; and the three triangles formed by lines drawn from D to the angular points, are equal to each other.

To determine the point D by CALCULATION, firſt find the area of ABC, which as performed by a well known rule, is 299·33259094: this divided by $\frac{1}{3}$CA gives 19·955506 = BH. Alſo ($\frac{1}{3}$ area ÷ by $\frac{1}{3}$CA) = 6·651835 = DG; ($\frac{1}{3}$ area ÷ by $\frac{1}{3}$AB) = 8·314794 = DI; ($\frac{1}{3}$ area ÷ by $\frac{1}{3}$BC) = 7·675194 = DL. Then, by reaſon of the parallels DP, BA, — DG, BH, we have, BH : DG (:: 3 : 1) :: BA : DP = 8. Again $\sqrt{(DP^2 - DG^2)}$ = 4·44444497 = GP; wherefore, GP + AP = AG = 14·444445 nearly. And $\sqrt{(GA^2 + GD^2)}$ = AD = 15·90248097, by which we have determined D, ſince AP and DP are known. The diſtances BD and CD may be found by means of 47, Euc. I. AB, BC, DI and DL, as well as AD, being known; as may be readily traced out, by thoſe who have leiſure and inclination to purſue the operation through all the requiſite ſteps.

N. B. To avoid the imputation of plagiariſm it may be proper to remark that the above conſtruction for the point D, is nearly ſimilar to one given in No. 11, of the Scientific Recepticle; but as the truth of the conſtruction is not demonſtrated in that work, I have here ſupplied the deficiency.

The ſame, anſwered by Merones Minor.

Fig. 510, Pl. 27. Biſect the ∠s A, B, C, of the given △ by the lines AE, BF, CD, and their interſection, O, will be the place of the weight. Then whether the weight be ſupported at A, B, and C, or only at A and E, the preſſure at A will be the ſame;

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ſame; and which, *by Em. Mech.* 31, *Cor.* 2. 800, is $= \frac{100EO}{AE}$ $= \frac{100BE}{AB+BE} = \frac{100BC}{AB+BC+AC}$, *ibid. Geom.* 25, *2nd and* 13. *Proportion*, = 30lbs. In like manner, the preſſure at B $= \frac{100AC}{AB+BC+AC} = 32\frac{1}{2}$lb, and that at C $= \frac{100AB}{AB+BC+CA}$ $= 37\frac{1}{2}$lb.

He alſo obſerves that in the ſecond caſe, as the preſſures are to be equal, the diſtances muſt be equal alſo; ſo that the centre of the circumſcribing circle will be the point required, and its diſtance from each of the angular points = 15·6344.

This queſtion was ingeniouſly anſwered by Meſſrs. Barron, Buſſham, Davis, Hill, May, *and* Lowry.

XV. QUESTION 225, *anſwered by Mr.* Cunliffe.

Multiplying the terms of the ſeries by x and taking the fluents, there will be had,

h. l. $(1+x)$+h.l.$(1+x^2)$ + h.l.$(1+x^4)$+h.l.$(1+x^8)$ &c. *ad infinitum*: put = h. l. (y); then by the nature of logarithms, $(1+x)\times(1+x^2)\times(1+x^4)\times(1+x^8)$ &c. $= y$; this laſt expreſſion being actually multiplied becomes $1+x+x^2+x^3+x^4$ &c. $= y$. But it is well known that $1+x+x^2+x^3+x^4$ &c. $= \frac{1}{1-x}$, and therefore $(1+x)\times(1+x^2)\times(1+x^4)\times(1+x^8)$ &c. $= \frac{1}{1-x}$. In which expreſſion, writing x^{2^n} in the place of x it becomes

$(1+x^{2^n})\times(1+x^{2^n+1})\times(1+x^{2^n+2})\times(1+x^{2^n+3})$ &c. $=$ $\frac{1}{1-x^{2^n}}$:

but $1+x^{2^n}$, is the $(n+1)$th factor of the foregoing continued product; therefore dividing the former expreſſion by the latter there will be had

$(1+x)\times(1+x^2)\times(1+x^4)\times(1+x^8)$ &c. to n factors $= \frac{1}{1-x}$

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$\frac{1 - x^{2^n}}{1 - x}$, whence by the property of logarithms,

h. l. $(1 + x)$ + h. l. $(1 + x^2)$ h. l. $(1 + x^4)$ &c. to n terms $=$ h. l. $(1 - x^{2^n})$ — h. l. $(1 - x)$.

Taking the fluxions of this expreſſion and dividing by $\dot{x}$ we get $\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8}$ &c. to n terms $= \frac{1}{1-x} - \frac{2^n \times x^{2^n - 1}}{1 - x^{2^n}}$.

And thus the queſtion was anſwered by Mr. Whalley. *Other anſwers were received from Meſſrs.* Gregory, Hill, Lowry, May, Peacock, *and* Thornoby.

XVI. QUESTION 226, *anſwered by Mr.* Gregory, *the propoſer.*

In order to ſolve this queſtion, let the radius of either wheel be denoted by r, its weight by w, the radius of either axle by d, and the weight ſuſpend from it by p: then, according to Prob. 15, pa. 40. *Banks's Treatiſe on Mills*, we have $\frac{dpr}{r^2w + d^2p}$ for the accelerating force at the circumference, that of gravity being 1; this being drawn into $16\frac{1}{12}$ produces ·2007280288 for the velocity of the circumference of the wheel: and, as $r : d$, or as $2 : \frac{1}{2} :: $ ·2007280288 : ·025091096, velocity of the circumference of the firſt axle, or ſpace paſſed over by the weight ſuſpended from it in the firſt ſecond; let this be denoted by a. Again, in the ſecond wheel, $\frac{·25 \times 5 \times 1·75}{(3·0625 \times 40) + (0625 \times 5)} =$ ·0178131543 is the accelerating force at the circumference; which drawn into $16\frac{1}{12}$ gives ·2864948984, velocity of the circumference of the wheel; alſo, as $r : d$, or as $1\frac{1}{4} : \frac{1}{4} :: $ ·2864948984 : ·04092784286 velocity of the circumference of the ſecond axle, or ſpace paſſed over by the weight ſuſpended from it in the firſt ſecond; let this be repreſented by b. Then, putting x for the ſpace deſcended by the firſt weight in the whole time, by the nature of deſcending bodies, we have $a : x :: 1^\circ : \frac{x}{a}$, ſquare of the whole time; and, by the queſtion

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queſtion $b : x + 50 :: 1^2 : \frac{x+50}{b}$, ſquare of the whole time alſo; therefore, $\frac{x}{a} = \frac{x+50}{b}$, whence $x = \frac{50a}{b-a} =$ $\frac{1.25455018}{.015836839} = 79.2165774$ feet. Conſequently, .025091003 6 : 79.21657724 :: 1^2 : 3157.17053422, the ſquare of the time, and 56.18870468 ſeconds, the time required.

Or the time might be found rather more conciſely, by ſaying, as .015836839 (the difference between the diſtances deſcribed by the weights in the firſt ſecond): 50 (the given difference) :: 1^2 : $(56.1887)^2$ which gives the time of deſcription as before.—— Other methods of calculation give 56.19 for the reſult, correſponding nearly with the above.

The ſame, anſwered by Miſs Suſan May.

Put W = weight of the greater wheel; = 50lb. and R = its radius = 2 ft.

w = weight of the leſſer wheel, = 40lb, and r = its radius = $1\frac{3}{4}$ ft.

a = rad. of the axle = $\frac{1}{2}$ ft. and b = weight ſuſpended therefrom = 5lb.

g = $16\frac{1}{12}$ feet, and t = time required.

Then, becauſe the weight of the wheels is ſuppoſed to be condenſed into their circumferences it follows from Prop. XIX. Sec. VI. of *Atwood's Treatiſe on Motion* that the accelerative force at the circumference of the axle of the greater wheel is = $\frac{a^2b}{WR^2 + a^2b} = \frac{1}{641}$ $(= p)$; and that the accelerative force at the circumference of the axle of the leſs wheel is = $\frac{a^2b}{wr^2 + a^2b} =$ $\frac{1}{349\frac{1}{3}}$ $(= q)$. And by Prop. IV. Sec. III. Cor. 5, *ibid*, the ſpace deſcended by the weight attached to the greater wheel is = $\frac{1}{2}gpt^2$, and the ſpace deſcended by the other weight is = $\frac{1}{2}gqt^2$. Therefore $\frac{1}{2}g(q-p)t^2 = 50$, or $t = \sqrt{\frac{50}{\frac{1}{2}g(q-p)}} = 56.19$ nearly, for the time required.

Ingenious ſolutions were alſo received from Meſſrs. Boſworth, Davis, Hill, Lowry, *and* Merones Minor.

XVII.

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XVII. QUESTION 227, *answered by Mr.* Cunliffe, *the proposer.*

Let ACB represent the triangular plot of land, with the perpendiculars DD, BF, and AE as per figure 512, plate 27.

Put $AC=a$, $AD=c$, $CB=b$, and $BD=d$. Then there will be had

$a^2-c^2=b^2-d^2=CD^2$, or $(a+c)\times(a-c)=r\times(b+d)\times(b-d)\div r=CD^2$;

put $a+c=r\times(b\times d)$ then $a-c=(b-d)\div r$; from these equations

$$a=\frac{r^2\times(b+d)+d-b}{2r}, \text{ and } c=\frac{r^2\times(b+d)+b-d}{2r}.$$

Now $b^2-d^2=CD^2$, must be a rational square;

therefore put $r\times(m^2+n^2)=b$, and $r\times(m^2-n^2)=d$, then will

$$a=\frac{r^2\times(b+d)+d-b}{2r}=r^2m^2+n^2; \text{ and } c=\frac{r^2\times(b+d)+b-d}{2r}=r^2m^2-n^2$$

from whence $AB=AD+DB=c+d=(r+1)\times(rm^2-n^2)$, where r, m, and n, may be taken at pleasure.

Therefore take $r=2$, $m=3$, and $n=2$; then $CB=r\times(m^2+n^2)=26$, $BD=r\times(m^2-n^2)=10$, $AC=r^2m^2+n^2=40$, and $AB=(r+1)\times(rm^2-n^2)=42$: but as each of these is divisible by 2, we may take $CB=13$, $BD=5$, $AC=20$, and $AB=21$.

Then by the similar triangles ACD, and ABF

$AC:AD::AB:AF=(AD\times AB)\div AC=(16\times 21)\div 20=16^4$,

whence $AC-AF=CF=8^1$. Also by the sim. $\triangle$s CBF, CAE, $CB:CF::CA:CE=(CF\times CA)\div CB=64\div 13$.

Wherefore, reducing the numerical values of AC, BC, FC, and EC to a common denominator, there will be had

$$AC=\frac{20.\ 13.\ 5}{13\ \ 5}=\frac{1300}{65}, \quad BC=\frac{13.\ 13.\ 5}{13.\ 5}=\frac{845}{65}, \text{ and } AB=\frac{21.\ 13.\ 5}{13.\ 5}=\frac{1365}{65}.$$

Therefore rejecting the common denominator 65, there may be taken $AC=1300$, $BC=845$, and $AB=1365$; and these are the numerical values of the sides of a triangle whose perpendiculars and segments of the sides made thereby are all whole numbers. The above numbers seem to be the least for an acute angled triangle, or one whose perpendiculars fall within it: but if we admit

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mit an obtuſe angled triangle the ſide AB′ will be 715, and is found by taking the ſegment DB′ from D towards A.

Theſe are the leaſt numbers of the kind that I have hitherto found; however they may not be the leaſt, as perhaps a different proceſs might diſcover others that are leſs. An iſoſceles triangle of leſs dimenſions might readily have been found that would anſwer; but that is not what was meant.

Meſſrs Gregory, Lowry, *and Miſs* May, *alſo favoured us with neat ſolutions to this queſtion.*

XVIII. QUESTION 228, *anſwered by Mr.* John Andrew, *Cark.*

Analysis. Fig. 513. Pl. 27. Suppoſe the thing done, B and H the given points, CD the line given in poſition, CHI the given angle, and BCHI the circle required, the centre of which is O. Join the given points B, H, with the ſtraight line BH, which biſect at A by the ⊥ AG, meeting CD in G, which is a given point, and AG a given line. From A and O, upon CD drop the ⊥s AP, OP′, and draw the radii CO, HO and IO, paral'el to which draw AD.

The ∠ COI = ∠ CHI (Euc. 20. III.), theref. the ∠ COP′ = P′OI = CHI = PAD, wheref. PAD is = the given ∠, and AD a given line. Moreover the right ∠d △s IP′O, DPA are equiangular, and ſo are the △s GOI, GAD; hence AG: AD :: GO: OI = OH: but AG and AD are given lines, theref. the ratio of OG: OH becomes known. Now, G and H being given points, the centre O may be found by the following

Construction. Having found the points A, G, and D, as in the *Analyſis*, join GH, and from A to GH produced, apply AQ=AD; alſo draw HO parallel to the line joining the points A and Q, to meet AG at O, the centre of the inſcribed circle.

The ſame, anſwered by Mr. Lowry.

Construction. Fig. 514. Pl. 27. Let A and B be the given points, and CD the ſtraight line given by poſition. Biſect the line adjoining the points A, B, at F, and draw FO ⊥ to AB meeting CD in O. On CD demit the ⊥ FH, and draw the FS to make the ∠ HFS = to the given ∠, and meet CD at S. In a ſemicircle deſcribed on the diameter AO, apply OQ = to a 4th proportional to SF, OF and AF. Draw AQ to meet FO at G; ſo ſhall G be the centre of the required circle.

Join

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Join A, G; G, D; A, D; and A, C; and on CD demit the ⊥ GI. It is evident that the △s AFG, GOQ are equiangular; theref. AG : GD :: AF : GO : OQ, but by conſtruction,

SF : OF :: AF : OQ; therefore, by equality,

GD : GO :: SF : OF; wherefore GD is parallel to SF. Conſequently the ∠ CAD = DGI = SFH = the given ∠ by conſt.

The same, answered by Mr. I. H. Swale.

Fig. 515, Pl. 27. Suppoſe the circle deſcribed through the given points A, B, as required, and cutting the indefinite right line CD in the points C, D. Join BA, BC, BD; draw DE ‖ to BA meeting the circle at E, and join CE.

Then, ſince the ∠ DBC = DEC, and ∠ EDC = BPC, are given, the ∠ DCE will alſo be given. Biſect AB at G, by the ⊥ FGH, meeting CD, DE in F and H, then will DH = HE; therefore DF : FC is a given ratio; for, from above, the △ DCE is given in ſpecies. Conſequently, ſince, the rect. CP·PD = AP·PB, and the line FP are given we have the following

CONSTRUCTION. Let the right line BA, joining the given points B and A, meet the indefinite right line CD, given in poſition, in P. Biſect BA in G and draw GF ⊥ to BA to meet CD at F; draw FK to BA making the ∠ FKP = the given one, and biſect PK at L. Draw PQ ⊥ to CD and = to the ſide of a ſquare = to the rect. BP·PA; join QF; make FI : FQ = LG : LP (LK), and on QI deſcribe a ſemicircle cutting CD in C; then through A, B and C, deſcribe a circle, and it is done.

DEMONSTRATION. Let the circle meet CD again in D; join BC, BD; IC, QC, QD; draw DE ‖ to BA to meet the circle at E, join CE, produce FG to meet DE at H, and let LR, drawn ‖ to GF, meet PF at R.

By conſtr. $PQ^2 = PB \cdot PA = PC \times PD$, and PQ ⊥ to CD; therefore the points D, Q, C are in a ſemicircle, or CQ ⊥ to QD. But by conſtr. the points I, C, Q, are in a ſemicircle, or CQ ⊥ to CI. theref. CI, DQ are parallel; and therefore

FC : FD = FI : FQ = LG : LB, by conſtr. = RF. : RP, by ‖s; alſo DH : HE = PL : LK, *i.e.* DC : PF = CE : FK, or CE is ‖ to FK. Hence, ∠ PKF = DEC = DBC by the circle = the given ∠ by conſtr.

The same, answered by Mr. James Whalley.

ANALYSIS. Fig. 516, Pl. 27. Suppoſe the circumference of the circle to paſs through the two given points E and A, and to cut the line FG given by poſition in the points F, and G,

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ſo that drawing FE, GE, the ∠ FEG may be of a given magnitude. Draw the right line EA cutting FG in I, and ſuppoſe GO to be a tangent to the circle at G meeting EA produced in O. Then, by Euc. 36. III. OA × OE = OG², and, by Euc. 32. III. the ∠ IGO = FEG = a given ∠, alſo the ∠ GIO is given from the poſition of FG, whence the △ IGO is given in ſpecies, and theref. the ratio of OG to OI, or of OG² = OA × OE to OI², is given, Now, E, I, and A are given points in the indefinite right line EO, theref. the point O may be found by Prob. 6, of *Wales's Determinate Section*, and belongs to fig. 20, of that problem.

Ingenious Solutions to this queſtion were also received from Messrs. Cunliffe, Hill, May *and* Peacock.

XIX. QUESTION 229, *answered by the proposer, Mr.* Robert Wallace.

Let EBHCGF (fig. 541, pl. 28.) be a paraboloid, and A a corpuſcle of matter ſituated in the axis produced. Put AD = y, AH = a, AB = z, $y - a$ = HD, theref. $y^2 + DC^2 = AC^2 = z^2$, $b\,(y - a) = DC^2$ (b, being the latus rectum), that is $by - ab = z^2 - y^2$, theref. $y^2 + by = z^2 + ab$, and theref. $y^2 + by + \frac{1}{4}b^2 = z^2 + ab + \frac{1}{4}b^2$; hence $y = \pm\sqrt{(z^2 + ab + \frac{1}{4}b^2)} - \frac{1}{2}b$, $\dot{y} = \frac{z\dot{z}}{\sqrt{(z^2 + ab + \frac{1}{4}b^2)}}$: put $ab + \frac{1}{4}b^2 = g^2$ then $\dot{y} = \frac{z\dot{z}}{\sqrt{(z^2 + g^2)}}$.

Now the attraction of all the particles in the circular ſurface BDC (vid. Simpſon's Fluxions, Vol. 2d. Art. 376,) is $= (AD \times AB^{n+1} - AD^{n+2}) \times [2P \div (n + 1)]$, P, being $= 3{\cdot}14159$, &c. Therefore by ſubſtituting and fluxing we obtain $(y\dot{y}z^{n+1} - y^{n+2}\,\dot{y}) \times [2P \div (n+1)]$

$$= [(\sqrt{z^2 + g^2} - \tfrac{1}{2}b) \times \frac{z\dot{z}}{\sqrt{(z^2 + g^2)}} \times z^{n+1} - y^{n+2}\,\dot{y}] \times \frac{2P}{n+1}$$

$$= [(z\dot{z} - \frac{bz\dot{z}}{2\sqrt{(z^2 + g^2)}}) \times z^{n+1} - y^{n+2}\,\dot{y}] \times \frac{2P}{n+1}$$

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$= (z^{n+2}\dot{z} - \frac{bz^{n+2}\dot{z}}{2\sqrt{(^2x-g^2)}} - y^{n+2}\dot{y}) \times \frac{2P}{n+1}$ = the fluxion of the whole force, the whole fluent of which, being found by *Simpson's first volume*, and corrected will give the whole force of the attraction of the paraboloid, &c.

XX. QUESTION 230, *answered by the proposer*, Mr. Cunliffe.

Denote the required fractions by $\frac{\frac{1}{4}x}{x+y}$ and $\frac{\frac{1}{4}y}{x+y}$; for upon trial it will be found that if either of them be added to the ſquare of the other, the ſum thence ariſing will be a rational ſquare. Therefore it only remains to determine ſuch values for x and y as ſhall make the ſum of the ſquares and the ſum of the cubes of the above fractions rational ſquares,

viz. $\frac{1}{16} \times \frac{x^2+y^2}{(x+y)^2}$ and $\frac{1}{64} \times \frac{x^3+y^3}{(x+y)^3} = \frac{1}{64} \times \frac{x^2-xy+y^2}{(x+y)^2}$ each a ſquare; or which amounts to the ſame thing, x^2+y^2 and x^2-xy+y^2, muſt each be a ſquare.

Put $2mn = x$ and $m^2-n^2 = y$, then $x^2+y^2 = (m^2+n^2)^2$, which is manifeſtly a ſquare: alſo $x^2-xy+y^2 = m^4-2m^3n+2m^2n^2+2mn^3+n^4$ = a ſquare. Aſſume m^2-mn+n^2 for its root, and then $m^4-2m^3n+2m^2n^2+2mn^3+n^4 = (m^2-mn+n^2)^2 = m^4-2m^3n+3m^2n^2-2mn^3+n^4$, which being reduced gives $m = 4n$; wherefore $x = 2mn = 8n^2$, and $y = m^2-n^2 = 15n^2$; whence $\frac{\frac{1}{4}x}{x+y} = \frac{8}{92}$ and $\frac{\frac{1}{4}y}{x+y} = \frac{15}{92}$, which are two fractions that will anſwer the conditions of the queſtion.

The ſame queſtion anſwered differently.

Let $\frac{1}{2} - x$ and x denote the two required fractions;—for it will be found that either of them being added to the ſquare of the other will make a rational ſquare.

It therefore only remains to find the value of x, ſo that the ſum of the ſquares, viz. $\frac{1}{16} - \frac{x}{2} + 2x^2$, and the ſum of the cubes,

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viz. $\frac{1}{64} - \frac{8x}{16} + \frac{32x^2}{64}$, of the two fractions may both be ſquare numbers; or, which is the ſame thing, that $1-8x+32x^2$ and $1-12x+48x^2$, may each be a ſquare number. Aſſume $4nx-1$ for the root of the former of theſe, that is, put $1-8x+32x^2 = (4nx-1)^2 = 16n^2x^2 - 8nx + 1$, which gives $x = \frac{n-1}{2(n^2-2)}$; this being written for x in the other expreſſion $1-12x+48x^2$, it becomes $1 - \frac{6(n-1)}{n^2-2} + \frac{12(n-1)^2}{(n^2-2)^2} = \frac{1}{(n^2-2)^2}$ $\times (n^4-6n^3+14n^2-12n+4)$, which muſt be a ſquare number by the queſtion, therefore $n^4-6n^3+14n^2-12n+4$ muſt be a ſquare. Aſſume n^2+3n-2 for its root, or make $n^4-6n^3+14n^2-12n+4 = (n^2+3n-2)^2 = n^4+6n^3+5n^2-12n+4$, which being reduced gives $n = \frac{1}{4}$; therefore $x = \frac{n-1}{2(n^2-2)} = \frac{1-n}{2(2-n^2)} = \frac{8}{56}$, and $\frac{1}{4} - x = \frac{1}{4} - \frac{8}{56} = \frac{15}{56}$, which are the very fractions obtained by the other method of ſolution.

Mr. Lowry *and Miſs* May *likewiſe ſent ſolutions to this queſtion.*

XXI. QUESTION 231, *anſwered by Mr.* James Whalley.

Fig. 517, Pl. 27. *Analyſis.* Let ABCD repreſent the trapez. AB the given ſide, P the given point therein, through which the ſide CD, oppoſite to the given ſide AB, paſſes; CAB the given $\angle$, and QE the right line given by poſition, in which is ſituated the angular point D oppoſite to the given $\angle$ A. Produce AC to meet QE in E, and draw BF ‖ to QE meeting AE in F and join FD, DB.

The area of the $\triangle$ FEB $=$ $\triangle$ FDB, by Euc. 37. I. therefr. $\triangle$ AEB $=$ trap. AFDB $=$ a given ſpace. But the area of the trap. ABCD is given by the queſtion. Wherefore the area of the $\triangle$ CFD becomes known. Whence the following

CONSTRUCTION *by Mr.* James Cunliffe.

Draw the right line PF, and produce AF, ſo that the area of the $\triangle$ FPS may be $=$ to the given ſpace CFD; through S draw an indefinite right line SN parallel to PF. Then through P draw

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draw the right line NPCD *by Prob.* 37, *Simpson's Geometry*, so that PN = CD, and the thing is done.

The same, answered by Mr. I. H. Swale.

Fig. 517, Pl. 27. Suppose the trap, ABCD determined as required, AB the given side, BAC the given ∠, P the given point in BA produced, DC the side passing through P, and DE the right line given in position, in which the ∠ BDC, or the point D is posited: Produce AC to meet DE at E and join EB meeting PD in Q', then will the point E be given, therefore the △ AEB will be given in magnitude: but the trap. ABCD is also given, theref. their difference, or the difference of the △s CQ'E, DQ'B will be given; theref. by adding common space DQ'E to each, the difference between the areas of the △s DEC, DEB will be given; and, since the base DE is common to those △s, the rectangle. under DE, and the difference of the ⊥s from C and B upon DE, will be given: Now, draw CG, BF ⊥ to DE, and BF ‖ to FD, meeting CG in H. Then, since BF = GH, it follows that CH × DE is a given rectangle. Draw BE' ‖ to AE, the △s BE'F, CHF are similar, theref. CF : CH = BE' : BF', a given ratio, wherefore the rect. CF × ED is given. Now E and F are given points, and ED, EA straight lines given by position, and P being also a given point, the problem is, therefore, reduced to the *sectione spatii* of the Ancients.

The same, answered by Mr. Lowry.

Fig. 518, Pl. 27. CONSTRUCTION. Let AB be the given side, P the given point, and ZL the straight line given by position. Draw AC to make the given ∠ with AB, and on AB constitute a parallelogram BAGK = to twice the given area. By Prob. 37 of *Simpson's Geometry*, draw PI to meet AC in C, ZL in D, and GK produced in I, so that CD may have to CI the given ratio of AB to PB. Join BD, and ABCD will be the trapezium required.

Through C and D let CE, VDF, be drawn ⊥ to AB, and ‖ to AB, let CQ be drawn, meeting DF in Q.

Then by reason of the parallels,

QD : QV :: CD : CI :: AB : PB, by construction; therefore, QD·PB = QV·AB, and, by adding CE·AB to each, we have. QD·BP + CE·AB = (QV + CE)·AB = the paral. BG. Again, PB·DF = PB·(EC + QD) = 2 △ PDB, and

PA·CE = CE·(PB − AB) = 2 △ PAC; therefore, their difference QD·BP + CE·AB = trap. ABCD = parallelogram BAGK.

Consequently the trapezium is equal to the area. The rest is evident from the construction.

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XXII. QUESTION 232, *anſwered by Mr.* Lowry.

Fig. 519, Pl. 28. Let ABCD be a trapezium inſcribed in the circle ABCDEF; draw the diameter FHE ⊥ to one of the diagonals as AC, and on the oppoſite ſides AB, DC, demit the ⊥s EL, FK. Draw, BG, DI, ‖ to AC, to meet EF at G and I, and join A, F; A, E; D, E; D, F; B, E; C, E; and CF.

Then by the ſimilar triangles AFE, LBE,
$AF^2 : EF^2 :: LB^2 : BE^2$, that is, (by Simpſon's Geom. Cor. to 19. VI.)
$FE \cdot FH : EF^2 :: LB^2 : FE \cdot EG$; therefore $LB^2 = GE \cdot FH$.
In a ſimilar way it is ſhewn, from the ſim. △s KDF, CEF, that $DK^2 = EH \cdot FI$, and, from the ſim. △s ALE, FGB, that $AL^2 = EH \cdot FG$; and alſo, from the ſim. △s CKF, EID, that $CK^2 = FH \cdot EI$. Therefore
$LB^2 - CK^2 = FH \cdot IG$, and $DK^2 - AL^2 = EH \cdot IG$.
Again, by the circle, FH : AH :: AH : EH, therefore
$FH \cdot IG : AH \cdot IG$ (= trapez. ABCD) :: $AH \cdot IG : EH \cdot IG$, or
* $LB^2 - CK^2$: trap. ABCD :: trap. ABCD : $DK^2 - AL^2$.

Now it is evident that LB is = to half the ſum of the ſides AB, BC, and DK = half the ſum of the ſides AD, DC; therefore LB + DK is = to half the perimeter, and conſeq. the exceſſes of half the perimeter above each of the ſides AB, BC, DC and AD, are DK — AL, DK + AL, LB — KC, and LB + KC, reſpectively. But by *cor.* 1. 9. II. *Simpſon's Geometry*,
$(LB - KC) \cdot (LB + KC) = LB^2 - KC^2$, and
$(DK - AL) \cdot (DK + AL) = DK^2 - AL^2$,
therefore it is manifeſt from what is proved above * that the trapezium is a mean proportional between the rectangle under the exceſſes of half the perimeter above two of the ſides, and rectangle under its exceſſes above the other two ſides, which is the rule alluded to in the queſtion.

The proportion marked thus * ſuggeſts another rule for finding the area, which will on many occaſions be found more commodious than that given above.

XXIII. QUESTION 233, *anſwered by Mr.* Cunliffe,

Fig. 520, Pl. 28. In any plane △ ACB inſcribe a circle, touching the baſe AB in T, and the other two ſides in Q and R. Draw the right line CI to biſect the baſe in I, and through the centre O of the circle draw the right line TP to meet CI in P;— through P draw *ab* parallel to AB, to meet the ſides AC, BC in *a* and *b* reſpectively; and draw O*a*, O*b*, OQ, OR.

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Since *ab* is parallel to AB it will be biſected by CI in P; or *a*P = P*b*. Now TP is ⊥, or at right angles, to AB, by Euc. 18. III; therefore PO is ⊥ to *ab*, by Euc. 28. I.; wheref. the △ *a*O*b* is iſoſceles, or *a*O = O*b*; but OR = OQ, and ∠ OQ*a* = OR*b* = a right ∠, by Euc. 18. III.; theref. ∠ *b*OR = *a*OQ, by Euc. 4. I.

Becauſe ∠ OP*b* = OR*b* = a right ∠, a circle will paſs through the points O, P, *b*, R, and therefore ∠ *b*OR = *b*PR, by Euc. 21. III. For the ſame reaſon a circle will paſs through the points O, P, Q, *a*, and therefore ∠ *a*OP = *a*PQ, conſequently ∠ *b*PR = ∠ *a*PQ; therefore the points Q, P, R are in the ſame ſtraight line, by Euc. 15. I.

The ſame, anſwered by Mr. Lowry.

Fig. 520, Pl. 28. Let ACB be the △, CI the line biſecting the baſe, O the centre of the inſcribed circle, OP a part of its diameter drawn from the point where it touches the baſe; and R and Q the other points of contact. About the △ deſcribe a circle, and let the diameter XIE be drawn ⊥ to the baſe; alſo let CE be joined, which will paſs through O, the centre of the inſcribed circle....Now, if the line joining the points R and Q meets OP at P, the truth of the propoſition will appear manifeſt if we can prove the points I, P, C to be in a ſtraight line. In order to do this, through I, draw GIH ‖ to RQ to meet the ſides at G and H, alſo draw AY ‖ to BC to meet IG produced at Y, and join GE. Then becauſe RC = CQ, CG will be = to GH, alſo AY is = AG = BH; theref. AG is = half the diff. of the ſides AC, CB; wheref. by a well known property, GE is ⊥ to AC, and conſequently ‖ to PO. Theref. the △s EGI, ROP are equi-angular, and

GI : GE :: RP : RO. alſo CG : GE :: CR : RO; wherefore,
GI : CG :: RP : CR. Therefore I, P, C, are in a ſtraight line.

If O′ be the centre of a circle touching the baſe and the continuation of the other two ſides in X′, R′, and Q′, then if O′X′ be drawn to meet IC produced in P′; the points R′, P′, Q′, will be in the ſame ſtraight line.

The ſame, anſwered by Mr. Swale.

Fig. 520, Pl. 28. Let ACB be a plane △, biſect the baſe AB at I by the ⊥ IX; alſo biſect the vertical ∠ ACB by the line CE, meeting IE in E: demit upon the ſides CA, CB, the ⊥s EG, EH, and join GH. Then, I ſay GH will paſs through I:—For

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I:—For, let GH meet EC in D, then since EC = EC, ∠ ECG = ECH, and ∠ EGC = EHC, the △s EGC, EHC are equal in all respects, theref. CG = CH, and theref. the △ GCH is an isosceles one. But CD or CE bisects the vertical ∠ GCH of the isosceles △ GCR, theref. GD is ⊥ to CE. Hence, by Harrison's Propositions in the Repository, GD passes through I, the middle of AB.

Now, take any other point O in EC, and demit upon CA, CB the ⊥s OR, OQ; draw OP ∥ to EI, meeting CI in P, and join RQ. Then, I say, RQ will pass through P:—For, let RQ, meet OC in D', and through P, ∥ to AB draw *ab*, meeting CA, CB at *a* and *b*; then since AI = IB, and *ab* ∥ to AB, the base *ab* will also be bisected by the line CI, that is, *a*P = P*b*. Now, by reasoning as above, it appears that QD' is ⊥ to CO, and also passes through P the middle of *ab*, therefore RQ passes through P.

Whence, taking the point O, the centre of the inscribed circle, the demonstration follows from the foregoing general one.

Cor. Produce PO to meet AB at T, produce also IE to H', making EH' = EG = EH: the points C, T, H' are in a right line.

XXIV. QUESTION 234, *answered by Mr.* I. H. Swale.

Fig. 521, Pl. 28. Produce FG, KH, to meet at Q, and join QA, QB, QC; let AL, BQ, CQ meet BC, CF, BK at I, S, R, respectively: the lines BK, CF intersecting at P.

Now, QG = AH = AC, GA = AB, and the ∠ AGQ = BAC, theref. the △s ABC, AGQ are equal in all respects, and theref. the ∠ GQA = BCA = BAI, that is, because GQ, BA are parallel, QAI is a continued right line, that is, IA produced passes through Q; theref. QI is ⊥ to BC.

Again, since, AC = CK, AQ = BC, ∠ BCA = QAH, and ACK, CAH right angles, it follows that the △s BCK, CAQ, are also equal in all respects, theref. ∠ CBR = CQI, that is, CRB is a right angle; theref. BR is ⊥ to QC.

In like manner it may be shewn that CS is ⊥ to QB. Hence, QI, BR, CS, are respectively ⊥ to the sides BC, CQ, BQ of the △ CBQ. But the ⊥s from the respective △s of any plane △, upon the opposite sides, are known to meet in the same point. That is, AL, BK, CF, intersect each other at the same point P.

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The same, answered by Meroles Minor.

Instead of AL being drawn ⊥ to BC, let it pass through the intersection P of BK, CF. Then AD appears to be ⊥ to CS, for ∠ CFB = BAD, *Emerson's Geom.* 6. II. and ∠ BcF = AcS' *Ibid.* 2. I; consequently ∠ AS'c = cBF = a right angle. In like manner it may be shewn that AE is ⊥ to BK. But, *ibid.* 24. II. The three ⊥s drawn from A, E, D, upon the opposite sides, all meet in one point M; theref. CP, BP being = and parallel to EM, DM, respectively, they must also meet AL in one point P, when AL is ⊥ to DE or BC.

The same, answered by Mr. Lowry.

The lines mentioned in the question do meet in the same point, as will be manifest from what follows. Let the lines CF, BK, meet in P, and through P draw APL to meet DE at L. Then it must be proved that AL is ⊥ to DE. Draw AD, AE meeting BK in R' and S', and CF in V and Q', and join the points R', V and S'Q'. It is evident from the proposition mentioned in the question that the ∠s BCS', BDS' are equal, and also the ∠s CEQ', CBQ', theref. the ∠s DS'C, BQ'E are right angles, and the points C, E, D, B, S', Q', are in the circumf. of a circle, and conseq. the ∠s Q'S' V, Q'BC, are equal. Again, because of the right angles at S' and Q', the points R', S', Q', V are in the circumf. of a circle, wheref. the ∠ K'R'V is = to the ∠ Q'S'N, that is = to the ∠ Q'BC; theref. R'V is ‖ to BC. Let AL meet R'V at N, then the ∠s R'VS', KQ'S, and PAS' are equal (for the points S', P, Q', A are in the circumf. of a circle), theref. the △s AS'P, PNV are equiangular, and AS'P is a right angle, theref. PNV is also a right angle, and consequently AN is ⊥ to R'N, and AL to DE.

Messrs. Hill, Marrat, May, *and* Peacock *favoured us with neat demonstrations of this proposition.*

XXV. QUESTION 235, *answered by Mr.* Cunliffe.

When the given point is within the circle.

Analysis. Fig. 522, Pl. 28. Let DPE be the required line, passing through the given point P and cutting the periphery of the circle in the points D and E. Through P draw the diameter AB, upon which let fall the perpendiculars *Dm*, *En*; and from the centre C draw CQ ⊥ to DE. Draw *Ee*, QR ‖ to AB, meeting

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meeting Dm in e and R respectively: also draw $Cr \parallel$ to Dm to meet QR in r.

CASE I. When $Dm + En =$ a given magnitude.

The $\triangle$s PCQ, and EDe are similar, theref. CP:CQ::DE:De, whence (Euc. 16. VI.) $CQ \times DE = CP \times De$; now CP is given, and $De = Dm + En$, is given by the question; from whence, therefore, CQ may be found. Moreover CQP being a right angle, the *locus* of Q will be a circle whose diameter CP is given:—if theref. this circle be described and CQ determined as above, this case of the problem may be easily constructed.

CASE II. When $Dm - En$ is a given magnitude.

The chord DE is bisected in Q (Euc. 3. III.), wherefore because of the parallels Ee, QR, De is bisected in R, and theref. $Rm = Cr =$ half the diff. of Dm and me, or of Dm and En. Whence this case may be readily constructed.

CASE III. When the rectangle $Dm \times En =$ a given magnitude.

Draw PT $\perp$ to AB, to meet the periphery of the circle in T. The $\triangle$s PDm, and PEn are similar, theref.

PD : PE :: PD^2 : PD $\times$ PE $=$ PT^2 :: Dm : En :: Dm : $Dm \times En$; from whence, by alternation, PD^2 : Dm^2 :: PT^2 : $Dm \times En$.

The $\triangle$s PDm and PCQ are also similar, wherefore

PD^2 : Dm^2 :: PC^2 : CQ^2. conseq. PT^2 : $Dm \times En$ :: PC^2 : CQ^2; now PT and PC are each given, and the rectangle $Dm \times En$ is given by the question; from whence, therefore, CQ becomes known and a construction therefore quite obvious.

Or, a construction to the third case of this part of the problem will be very obvious from what follows.

Draw TV $\perp$ to DE; then, by the sim. $\triangle$s CQP, and PVT, PT^2 : PV^2 :: PC^2 : CQ^2, whence, and from what is done before, it appears that PV$^2 = Dm \times En$, *which is a curious theorem.*

When the given point is without the circle. Fig. 523, Pl. 28.

CASE I. When $Dm + En =$ a given magnitude.

The chord DE is bisected in Q (Euc. 3. III.), wherefore because of the parallels Ee, QR, DR $=$ Re, whence $2mR = 2Cr = Dm + em = Dm + En =$ a given magnitude by the question. And because CQP is a right angle, the *locus* of Q will be a circle whose diameter is CP:—theref. let this circle be described, and the following *construction* offers itself.

Erect $Cr \perp$ to AB and $=$ to half the given sum of Dm and En; through r draw $rQ \parallel$ to AB, to cut the periphery of the circle whose diameter is CP in Q; through P and Q draw PED cutting.

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cutting the given circle in E and D, and the thing is done as is evident from the analysis.

CASE II. When $Dm - En =$ a given magnitude.

It is evident that $Dm - En = Dm - me = De$ a given magnitude by the question. Moreover the $\triangle$s PCQ, and EDe are similar, theref. CP : CQ :: DE : De, whence (Euc. 16. VI.) $CQ \times DE = CP \times De$: but CP is given, and De is given by the question, theref. CQ may be determined from hence, and this case of the question thereby readily constructed.

CASE III. When the rectangle $Dm \times En =$ a given magnitude.

Draw the tangent PT :—Then by the parallels Dm, En, $DP:PE::DP^2:DP \times PE = PT^2:Dm:En::Dm^2:Dm \times En$; from whence by alternation $DP^2 : Dm^2 : : PT^2 : Dm \times En$.

The $\triangle$s PDm, and PCQ are similar, therefore $DP^2:Dm^2.:PC^2:CQ^2$, consequently $PT^2:Dm \times En::PC^2:CQ^2$; but PT and PC are each given, and the rectangle $Dm \times En$ is given by the question; whence CQ becomes known and the construction from thence manifest.

Or, a construction of the third case of this part of the problem may be had from what follows.

On PC take PT′ = PT, and upon PT′ as a diameter describe a circle cutting the line PD in V, and join T′V. The $\triangle$s PCQ, and PT′V are similar, theref. $PT'^2 = PT^2 : T'V^2 : : PC^2 : CQ^2$; wheref. from what is done above, it appears that $T'V^2 = Dm \times En$, *which is another curious theorem.*

The same, answered by Mr. Lowry.

When the given point is without the circle. Fig. 524, Pl. 28.

Let P be the given point, BEFA the given circle, and suppose the chord PEF to be drawn as required, namely, so that demitting on the diameter PBA the $\perp$s EH, FK, their sum difference, or rectangle may be given.

1. When the sum is given. Bisect EF at G, and on AB drop the $\perp$ GI; then GI is evidently = to half the sum of EH, FK, theref. GI is given. Now, if O be the centre of the circle, OG being joined will be $\perp$ to EF; theref. the *locus* of the point G will be a semicircle described on PO; consequently if PQ be drawn $\perp$ to PA and = to half the given sum, a parallel to PA, drawn from Q, will interfect the semicircle at G, through which if PG be drawn it will be done.

If E′ be the point where the circles interfect, PE′ will be a tangent to the circle BEF at E′; theref, it is evident that the given sum must not be greater than twice the $\perp$ drawn from E′ to AB.

2. When

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2. When the difference is given. In AB find the point P′ ſo that BP′ : AP′ :: PB : PA; then it follows from Lawſon's 7th Prop. (demonſtrated in the Repoſitory) that if any line whatever as PEF be drawn to meet the circle at E and F, and FP′ be drawn to meet the circle again at S, the line joining ES will be ⊥ to AB; conſeq. SH will be = to EH, and theref. the difference between FK and SH will be given. Biſect SF at V, and on AB demit the ⊥ VL, then it is evident that VL is = to half the diff. of FK and SH, and is therefore given. Now, becauſe OV is ⊥ to SF, the *locus* of the point V is a ſemicircle deſcribed on OP′. Theref. if in this ſemicircle VL be applied perpendicularly to P′O, and of the given length, the point V will be determined, and the remaining part of the conſtruction manifeſt.

The difference will evidently be a maximum when P′O is biſected at L in which caſe it is = P′O. Or if P′E′ be drawn, and OE′ joined; by Lawſon's 15th Prop. P′B′ is ⊥ to AB; theref. $OB^2 = OE'^2 = OP' \times OP$, and $OG^2 = PO \times OI$, (Euc. Cor. 8. VI.); wherefore $OE'^2 : OG^2 :: 2 : 1$.

3. When the rectangle is given.

By ſimilar triagles PE : PF :: EH : FK, and therefore,
$PE \times PF = PE'^2 = PO \times PP' : PF^2 :: EH \cdot FK : FK^2$; but by ſim. $\triangle$s, $PF^2 : FK^2 :: PO^2 : OG^2 = PO \times OI$, therefore by equality,
$PO \times PP' : PO^2 :: EH \times FK : PO \times OI$, or
$PP' \times OI = EH \times FK =$ a given ſpace: but PP′ is given, theref. OI is given, and the method of conſtruction is evident.

The rectangle will manifeſtly be a maximum when OI is ſo; that is, when it be becomes = to OP′, or when EH and FK coincide.

When the given point is within the circle.

If P′ be the given point within the circle, and a point P be found without ſo that BP : AP :: P′B : P′A, the conſtructions and determinations will be the very ſame as the preceding.

The ſame, anſwered by Mr. I. H. Swale.

1. When the ſum of the perpendiculars is given and the given point within the circle Fig. 524.

Let P′ be the given point, draw the diameter BP′A, and P′E′ ⊥ thereto, meeting the circle at E′, and let the tangent E′P meet P′B produced at P. Perpendicular to BA, in a ſemicircle on OP, apply IG = half the given ſum; draw PG meeting the circle at E and F, and through E, P′, or F, P′, draw the required chord EP′B′, or FP′S.

Draw ES, FB′, meeting BA in H, K, and join OG: then ſince ES, FB′ are ⊥ to AB, we ſhall have HE = HS, and KB

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KB = CF, or HS + BK = HE + KF =, becauſe OG is ⊥ to EF, to 2IG = the given ſum by conſtruction.

When the given point is without the circle, Mr. Swale's *Conſtruction* is the ſame as Mr. Lowry's given above.

II. When the difference of the perpendiculars is given and the given point without the circle. Fig. 524.

Draw to the circle, the tangent PE′, and, demit upon AB, the diameter drawn through, the ⊥ E′P′. Perpendicular to BA, in a ſemicircle deſcribed on P′O, apply LV = half the given diff. and join P′V which produce to the circle at F, and through P, F, draw the required line meeting the circle at E and F.

Produce FP′ to the circle at S and join SE, meeting AB in H. Demit upon AB the ⊥ FK, alſo join OV. Then ſince ES will cut AB perpendicularly at H, we ſhall have EH = HS, and FK — EH = FK — HS. Now draw through V, parallel to AB, the line RVN, meeting EH, FK in R, N. The points P′, V, O, are in a ſemicircle, theref. OV is ⊥ to SF, that is, SV = VF; theref. by parallels FN = SR, FK = FN + NK = SR + NK = SR + RH = SH + 2HR = SH + 2LV = EH + 2LV, and FK — EH = EH + 2LV — EH = 2LV = the given diff. by conſtruction.

When the point is within the circle as at P′, it will be determined by the conſt. given above, the demon. being of courſe the ſame.

III. When the rectangle of the perpendiculars is given, and the point without the circle. Fig. 524.

Draw PBA, PE′, and E′P′ as in the laſt caſe. Apply in the circle, ⊥ to AB, L″L′ the ſide of a ſquare = to the given magnitude and to a ſemicircle on P′B, apply OS′ = OL″; join P′S′, produce P′S′ to the circle at E, and through PE draw the required line meeting the circle E and F.

Produce EP′ to the circle at B′, join B′F, meeting AB in K; upon AB demit the ⊥ EH, join OS′ which produce both ways to the circle at G′ and H. Then, as before, from known properties B′F is ⊥ to AB, and FK = KB′, or 2FK × 2EH = B′F × 2EH =, by ſim. △s, to B′E² — EF²: But ſince the points P′, S′, B, are in a ſemicircle, BS′ will be ⊥ to EB′; or B′E + EF = 2B′S′, and B′E — EF = 2ES, that is, 2B′S′ × 2ES′ = B′E² — EF² = B′F × 2EH = 2FK × 2EC; and FK × EH = B′S′ × S′E =, by the circle, to G′S′ × S′H′: Now, by conſtr. OS′ = OL″, and by the circle OB = OA = OG′ = OH′, or G′S′ = AL″, and H′S′ = BL″; that is, BL″ × L″A = G′S′ × S′H′ = FK × EH from above, = L″L′², by the circle, = the given magnitude by conſtruction.

When the point is within the circle, the conſt. will be the ſame as the above.

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Meſſrs. Andrew, Thomoby, *and* Whalley *alſo anſwered this queſtion very ingeniouſly.*

XXVI. QUESTION 236, *anſwered by the propoſer, Mr.* Lowry.

CONSTRUCTION. Fig. 525, Pl. 28. Let AB, BC and AC be the three ſtraight lines given by poſition, and let C*a*, C*b*, C*d*, be reſpectively ‖ to the lines drawn from the required point to AB, BC, and AC. Produce *a*C till it meets AB at D', and take D'*a* = to the given ſum; draw Q'*a*Q ‖ to AB to meet AC, BC, produced, in S and X. Take C*b* and C*d*, each = to C*a*, and draw S*b*M, X*d*N, to meet BC and AC, in M and N reſpectively; draw alſo MH ‖ to C*b*, to meet AC at H, and LN ‖ to C*d* to meet CB at L; join LH meeting AB, produced at R. Through H draw HIW ‖ to C*a*, and continue it to meet LW, drawn ‖ to AB, cut W. Then, by Prob. 2nd, of the Determinate Section, by Snellius, at HR at P, ſo that the rectangle HP·PL, may have to the rectangle contained by PR and the given line, the given ratio of HW·HL to LN·HM; then P is the point required.

DEMONSTRATION. From P draw PDG, PE, and PF ‖ to C*a*, C*d*, and C*b* reſpectively, and let the line drawn through the points I, L, meet GP at O, alſo draw LK ‖ to C*a*.

Then, becauſe of the parallels, it is evident, ſince C*a* (= C*b*) is = C*d*, that LK is = to LN, and HI = HM. And by ſim. △s,

IL : IO : : LK : QG (: : LH : HP) : : LN : PE,
but LK is = LN, therefore OG is = PE. Moreover,
IL : OL : : HL : LP : : HM : PF : : HI : PO,
but HI is = HM, therefore PO is = PF. Conſequently,
PE + PF + PD = D'*a*, that is, = the given ſum, by conſtr.
Again, let Q be the given line. Then by ſim. △s,
HP:PE::LH:LN, and PL:PF::LH:HM. and compounding
HP·PL:PE·PF::LH²:LN·HM. But, by conſtruction,
HP·PL:PR·Q::HW·HL:LN·HM, therefore, *ex æquo perturbate*,
PE·PF:PR·Q::HW·HL:LH²::HW:HL; and by ſim. △s,
HW : LH : : DP : PR : : DP·Q : PR·Q; therefore
PE·PF:PR·Q::DP·Q:PR·Q; and theref. PE·PF = DP·Q.

The ſame, anſwered by Mr. Cunliffe.

Lemma. To find a point, in one of three right lines, given by poſition, whence if two right lines be drawn in given angles to the other two, the ſum of the two lines ſo drawn may be of a given length.

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CONSTRUCTION. Fig. 526, Pl. 28, Let AB, AC and BC be the three lines given by position: Take AM of the given length and making with AB the given angle MAB; through M draw M*ba* parallel to AB, meeting BC and AC in *b* and *a* respectively. Draw AV to meet CB in V, making the ∠ AVC equal to that made with the line BC: upon AV take A*m* = AM, and draw *am* cutting BC in G; through G draw GD parallel to AV meeting AC in D, and the thing is done.

DEMONSTRATION. Through D, draw *g*L ‖ to AM, terminating in M*a* and AB in *g* and L. By reason of the parallels D*g*, AM; DG, A*m*; *a*A : AM = A*m* : : *a*D : D*g* = DG; therefore DL + DG = DL + D*g* = L*g* = the given length by constr. and Euc. 34. I. Moreover, ∠ DLB = MAB, and ∠ DGC = AVC, the given angles by construction and Euc. 28. I.

Q. E. D.

The above being premised, the solution of the problem may be as follows;

Fig. 527, Pl. 28. Let AB, AC and BC be the three lines given by position. In one of them as AC, by the preceding Lemma, find the point D, such, that drawing DG to BC, and DL to AB in the given angles of the straight lines to be drawn to these lines, DL + DG may be equal to the given sum of the three lines. In like manner, in BC find the point E, from whence drawing EH to AC, and EK to AB in the respective given angles of the lines to be drawn to these lines, EH + EK may also be equal to the given sum of the three lines. Through D and E draw the right line DEF to meet AB in F, and this line DEF is the *locus* of the required point P.

For, upon LD and KE take L*g* and K*h*, each equal to the given sum of the three lines, and draw D*h* and *gh*. From any point P in DF draw the lines PQ, PR, PS respectively parallel to DG, EH, DL to meet the lines BC, AC, AB in Q, R and S; also produce SP to meet D*h* and *gh* in *r* and *q* respectively. Because of the parallels PR, EH; PR, E*h*;

DE : DP :: EH = E*h* : PR = P*r*. Also

ED : EP :: *h*D : *hr* :: DG = D*g* : PQ = *rq*; wherefore

PS + PR + PQ = PS + P*r* + *rq* = S*q* = L*g* the given sum by construction.

Now, suppose P to be the point required, upon DF take D*d* equal to the given line, the rectangle under which and PS, shall be equal to the rectangle under PR and PQ, that is D*d* × PS = PR × PQ, and through *d* draw *de* parallel to AB meeting DL in *e*. Then by the sim. △s D*de* and PFS,

D*d* : D*e* :: FP : PS, whence D*d* × PS = D*e* × FP.

Also by reason of the parallels DG, PQ; EH, PR,

DE : DG : : EP : PQ, and DE : EH : : DP : PR, whence

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$DE^2 : EH \times DG :: DP \times EP : PR \times PQ = Dd \times PS = De \times FP$, a given ratio, becauſe DE, EH and DG are all given as alſo De; from whence it is plain that this part of the problem belongs to the determinate ſection, by means of which the point P may be found.

Mr. Swale *anſwered it.*

XXVII. QUESTION 237, *anſwered by Mr.* Cunliffe.

Put $P = 1000$, and let n denote the time in quarters, in which the debt will be liquidated, in the manner ſtated in the problem; alſo let r be taken ſuch that $r^4 = 1{\cdot}05$, whence $r = 1{\cdot}012272$, which is the amount of 1l. at the end of the firſt quarter at the given rate. Then by the queſtion, and the principles of compound intereſt, $\frac{1}{r}$ will expreſs the preſent worth of the firſt quarterly payment; $\frac{2}{r^2}$ that of the ſecond quarterly payment; $\frac{3}{r^3}$ that of the third quarterly payment; and, generally, $\frac{n}{r^n}$ will expreſs the preſent worth of the nth quarterly payment: wherefore

$$P = \frac{1}{r} + \frac{2}{r^2} + \frac{3}{r^3} + \frac{4}{r^4} + \&c.\ldots\ldots \frac{n}{r^n}.$$

The ſum of n terms of this ſeries may be obtained as follows.

By diviſion $\frac{x}{1-x} = x + x^2 + x^3 + x^4$ &c. taking the fluxions and dividing by $\frac{x}{x}$ gives $\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4$ &c. Again by diviſion $\frac{x^{n+1}}{1-x} = x^{n+1} + x^{n+2} + x^{n+3}$ &c. taking the fluxions and dividing by $\frac{x}{x}$ as before gives

$$\frac{(n+1)x^{n+1} - nx^{n+2}}{(1-x)^2} = (n+1)x^{n+1} + (n+2)x^{n+2} + (n+3)x^{n+3}\ \&c.$$

Now

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Now this latter feries is evidently a continuation of the former after the n leading terms: therefore taking the difference of the expreſſions there will be had

$$\frac{x+nx^{n+2}-(n+1)x^{n+1}}{(1-x)^2}=x+2x^2+3x^3+4x^4\ldots\ldots+nx^n.$$

Write $\frac{1}{r}$ in the place of x, and the laſt expreſſion will become

$$\frac{\frac{1}{r}+\frac{n}{r^{n+2}}-\frac{n+1}{r^{n+1}}}{(1-\frac{1}{r})^2}=\frac{r^{n+1}+n-(n+1)r}{r^{n+2}\times(1-\frac{1}{r})^2}=\frac{r^{n+1}+n-(n+1)r}{r^n\times(r-1)^2}$$

$$=\frac{1}{r}+\frac{2}{r^2}+\frac{3}{r^3}+\frac{4}{r^4}\ldots\ldots+\frac{n}{r^n}: \text{ Conſeq. } \frac{r^{n+1}+n-(n+1)r}{r^n\times(r-1)^2}=P;$$

whence by reduction $r^n=\frac{(n+1)r-n}{r-P\times(r-1)^2}$; and taking the logarithms;

$n\times\log.(r)=\log.(\frac{(n+1)r-n}{r-P\times(r-1)^2})$ or $n=\log.(\frac{(n+1)r-n}{r-P\times(r-1)^2})\div\log.(r)=$

$\log.(\frac{(n+1)r-n}{\cdot 86167})\div\cdot 0052972$; this formula is very convenient for finding the value of n by the rule of double poſition, by means of a few trials of which, there will be had $n=55\cdot 228$ quarters $=13\cdot 807$ years, which is very near the truth.

The ſame, anſwered by Mr. David Davis.

Put $p=1000$, $r=\sqrt[4]{1\cdot 05}$, the amount of 1£. at the end of the firſt quarter, and $t=$ the number of quarters required.

Then by the principles of compound interest and annuities, and the nature of the queſtion,

$pr^t=r^{t-1}+2r^{t-2}+3r^{t-3}+4r^{t-4}+$ &c. to t terms; therefore

$p=r^{-1}+2r^{-2}+3r^{-3}+4r^{-4}+$, &c. to t terms,

$=\frac{1}{r}+\frac{2}{r^2}+\frac{3}{r^3}+\frac{4}{r^4}+$ &c. to t terms,

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$= \frac{r^{t+1} - (t+1)\,r + t}{r^t \times (r-1)^2}$. See *Clarke's* Tranſlation of Lorgna's Series, Form 190, or *Dodſon's* Mathematical Repoſitory, Page 65, vol. ii. Whence $t = \log. \left(\frac{(t+1)r-t}{r-1(r-1)^2}\right) \div \log. (r)$, and $t = 55{\cdot}228$ the number of quarters required.

And thus nearly was the anſwer given by Merones Minor, *and Mr.* Marrat *of Boſton.*

To XXVIII. QUESTION 238, *we have received no anſwer.*

XXIX. QUESTION 239, *anſwered by Mr.* Lowry, *the propoſer,*

Analysis. Join BP (fig. 528, pl. 28.) and draw PG to make the $\angle$ BPG = the given one; take PG = PG and draw GI to make the $\angle$ PGI = BPE: then, it is evident that the $\triangle$s GPI, BPE, are every way equal, theref. GI is = BE. Take PH : PG :: M : N, and make the rectangle N × P = the given one, and draw HK $\parallel$ to GI meeting AC at L; then, KH : GI :: M : N, wherefore N × KH = M × GI = M × BE, and N × KH $\pm$ N × CF = N × P, or KH $\pm$ CF = P = a given line.

Moreover LA and LC are given lines, theref. KL $\pm$ FL is given, and PF may be drawn as in my ſolution to Queſtion 220. It may alſo be drawn differently as follows: Through the points K, L, F (fig. 529, pl. 28.) deſcribe a circle and draw the diameter TID $\perp$ to KF, alſo draw DR and TS $\perp$ to LF, and RQ $\perp$ to the line joining the points D, L; draw alſo SI $\parallel$ to D L. Then it is well known that LR is = to half the ſum, and LS = to half the difference of KL, LF, and that RQ and SI meet at I, the middle of KF; we have theref. the following ſimple *Conſtruction.*

1. When the ſum is given. Take LR = half that ſum, and draw LD to biſect the $\angle$ KLF; on LD drop the $\perp$ RQ, and make RD $\perp$ to LR; then a ſemicircle deſcribed on the diameter PD will interſect RQ at I.

2. When the difference is given take LS = half the given difference, draw ST $\perp$ to LS, meeting LT, drawn $\perp$ to LD, at T, and draw SI $\parallel$ to LD; then a ſemicircle deſcribed on the diameter PT will interſect SI at I.

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The ſame, anſwered by Mr. James Cunliffe.

CASE I. When the ſum of the rectangles BD × M and CE × N is equal to a given ſpace. Fig. 530, Pl. 28.

Join PB, and draw PF meeting AC in F, and making the ∠ BPF = to the given ∠. In PF take P*b* = PB and divide P*b* in R, ſo that N : M :: P*b* : PR. Alſo take RS of ſuch a length that the rectangle RS × N, may be = to the given ſpace, and the ∠ *b*RS = PBA. Then, by *Halley's Apollon. de Sectione Rationis*, from the given point P draw the right line PEG, cutting AC in E and RS in G, ſo that CE = GS; alſo draw PD to AB, ſo that the ∠ EPD = FPB and the thing is done.

For, draw *bd* ‖ to RS, meeting PE in *d*. By the conſt. ∠ PRS = PBD, Euc. 27. I. = P*bd*; alſo *b*PB = *d*PD, and taking away the common ∠ *d*PB, there will remain ∠ *b*P*d* = BPD: whence, by Euc. 26. I. the △s *b*P*d* and BPD are equal in every reſpect; wherefore, by reaſon of the parallels *bd*, RG; N : M :: P*b* : PR :: *bd* = BD : RG, theref. by Euc. 16, VI. RG × N = BD × M. By the conſtr. RG + GS = RG + CE = RS; wherefore RG × N + CE × N = BD × M + CE × N = RS × N, the given ſpace, by the conſtruction.

CASE II. When the difference of the rectangles BD × M and CE × N is equal to a given ſpace.

Take RS′ on the contrary ſide of P*b*, and of ſuch a length that the rectangle KS′ × N may be equal to the given ſpace. Then, by *Halley's Apollon. De Sectione Rationis*, draw the right line PEG cutting AC and RS′ in G ſo that CE = GS′; alſo draw PD to AB ſo that ∠ EPD = BPF, and the thing is done.

For, by what is done in the preceding caſe BD × M = CE × N, and by conſtr. GS′ — GR = CE — GR = RS′; wherefore CE × N — GR × N = CE × N — BD × M = RS′ × N the given ſpace by the conſtruction.

The ſame, anſwered by Mr. I. H. Swale.

CASE I. When the ſum of the rectangles is given.

Fig. 531, Pl. 28. The lines PD, PE being ſuppoſed drawn as required, join BC: Take BF = M, and CG = N, join FC, GB, and draw, to BC, the lines DH, EI, making ∠ BDH = BCF, and ∠ CEI = CBG, and produce DH, EI to meet at Q. The points D, C, F, H, are in a circle, and ſo are the points E, B, G, I, theref. BF × BD = BC × BH, and CG × CE = CB × CI. Theref. the ſum of the rectangles BD × BF, and CE × CG, that is, the ſum of the rectangles BD × M and CE × N is = to the ſum of the rectangles BC × BH, and BC × CI,

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CI, that is, = to the rectangle under BC and the sum of BH, CI, = to a given rectangle: But BC is a given line, therefore the sum of BH, CI is given; theref. HI is a given line. Now, through Q, draw RS ‖ to BC, meeting AB, AC in R and S; draw also RK, SN ‖ to QE, QD, meeting AC, AB, in K and N.

Now, ∠ QHI = BFC, and QIH = CGB, being given, the the △ HQI is given in species, and, being described upon a given base HI, it is therefore given in magnitude. Theref. since HQ: BR = DH: DB, is a given ratio, BR is given, and of course the line RS is given in position.

Moreover, by reason of the parallels, RD: DN = QR: QS = EK: ES, *et componendo*, RN: SK: = RD: EK, a given ratio; since the points R, S, are given from above, and the lines RK, SN, making given angles with RS, are given by position, or the lines RN, SK, and their ratio therefore given.

Now the lines AR, AK, are given by position, R and K are given points therein, and P is a given point without them; we have only therefore to draw two lines PD, PE, to contain a given ∠, meeting AR, AK, in D and E, and making DR: EK a given ratio, which has been done at question 152.

Case II. When the difference of the rectangles is given.

Fig. 532, Pl. 28. Having drawn lines as in the preceding case, it will appear, that, the rectangle under BC and the diff. of BH, CI is given, and because BC is a given line, the diff. of BH, CI will be given. Bisect BC in M, make IW = HM, and draw MV, WV parallel to DQ, EQ, meeting each other at V, and the lines AB, AC in L and K; through V draw RS ‖ to BC, meeting AB, AC in R, S, and DQ, EQ in U and T..... Then, the diff. of BH, CI will be = to the diff. of MI, MH, that is equal to MW, a given line; theref. the point W is given; conseq. the △ MVW is given in species, magnitude, and position, that is, the point V is given; therefore the line RS passes through a given point and is ‖ to a given line; consequently the lines KS, LN are given in length, N being the point where SN, drawn ‖ to VL, meets AB. Whence, by parallels, DL: LN = VU (= VT): VS = EK: KS; et alternando, LN: SK = DL: EK = given ratio: But AL, AK, are lines given by position; L, K, given points therein, and P a given point without them, DPE a given ∠, and LD: KE a given ratio. And therefore this case is the same as the preceding one.

Remark. When PD, PE, are to be drawn including a given angle, and making the sum or diff. of BD, CE = to a given line, the same conclusions are still derived.

For,

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For, accordingly as the ſum or diff is given, ſet off, above or below C (fig. to Caſe I.) CT or CV equal thereto. Then in the former caſe we ſhall have BD = TE, and in the latter BD = VE: theref. drawing, by the ſolutions to Queſtion 152, the lines PD, PE, making DPE = the given ∠, and BD : TE, or BD : VE in the ratio of equallity, and it will be done. The points B, and T or V being evidently given ones.

XXX. *Or*, PRIZE QUESTION 240, *anſwered by Mr.* Swale.

Let ABCD (fig. 533, pl. 28.) be the circle given in magnitude and poſition, and P, Q, and R the given points.

Suppoſe the required trapezium inſcribed in the given circle, and that the ſides CA, BD; AD, CB, when produced interſect each other at the points P and Q, alſo that one of the diagonals, as CD, paſſes the given point R.

Now, join PQ, and it will be given in length and poſition. From A and B draw AK, BL, parallel to PQ meeting the circle at K and L; then by parallels, the arches AL, BK, are equal. Join AL, and produce LA to meet PQ at G; then, again, by reaſon of the parallels, the ∠ AGP = ALB = ACB, by the circle; therefore the ∠ QGA = ſupp. of the ∠ QCA, or the points Q, C, A, G, are in a circle; therefore, PQ × PG = PC × PA. Again the ∠ PGA = PDA, therefore the points P, G, D, A, are in a circle; theref. GQ × QP = DQ × AQ = BQ × CQ; and therefore PQ × PG + PQ × QG = PQ² = PC × PA + QB × BC = the ſum of the ſquares of the tangents drawn to the circle from P and Q*. Wherefore it appears that the given points muſt be ſo ſituated, with reſpect to the circle, that the ſquare of PQ muſt be equal to the ſum of the ſquares of the tangents from P and Q, ſo that the problem *cannot* be conſtructed *generally* for any ſituation of the points, but only for the particular one ſpecified above.

Let O be the centre of the given circle, draw GO meeting the circle in I and H, and CD in S, alſo let CG when drawn meet the circle in F; then, becauſe PQ × PG = PC × PA, and GQ × QP = BQ × QC, it is evident, from Mr. *Harris's* demonſtration to *Lawſon's* 33rd propoſition, that GO is perpendicular to PQ, AK and BL, therefore the arch AK = the arch IK, and the points G, K, B, are therefore in a ſtraight line. Again, becauſe OG is perpendicular to PQ, by *Lawſon's* 11th propoſition IG × GH + GP² = AP × PC, that is, = GP × PQ; take GP²

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GP² from each, and IG × GH = PG × GQ; therefore, by *Lawſon's* 28th propoſition IG : GH :: SI : SH; conſequently S is a given point: Hence the following ſimple

CONSTRUCTION. Join the firſt two given points P, Q; and upon PQ, from the centre O of the given circle, demit the perpendicular OG, meeting the circle in H and I; make HS : SI = HG : GI, and join the third given point R with the point S; let RS meet the circle in C and D, join PD, QD, meeting the circle again at B and A, and join CA, CB: ſo ſhall CA, CB, produced paſs through the points P and Q, and ABCD will be the trapezium required, as is evident enough from the analyſis.

The truth of queſtion 10 in the Britiſh Magazine is evident from the expreſſion marked *

The following THEOREM is alſo true, and may be of uſe in the conſtruction of Problems.

THEOREM. Let S be the interſection of the diagonals of a trapezium, inſcribed in a circle, whoſe centre is O, join OS, and make OG equal to a third proportional to OS and the radius of the circle: Then the line joining the points of interſection of the oppoſite ſides of the trapezium produced, will paſs through the point G and be perpendicular to GO.

The ſame, anſwered by Mr. Richard Nicholſon, *Liverpool.*

ANALYSIS. Suppoſe the thing done, ABCD (fig. 540, pl. 28.) the trapezium required, and E, F and P the given points. Draw EF, and CH parallel thereto meeting the circle at H, draw alſo HAG meeting EF in G, and join GC. Now (Euc. III. 21, and I. 29.) the ∠ ABC = AHC = HGE; theref. a circle will paſs through the points A, B, F and G: wheref. (Euc. III. 36.) the rect. EF·EG = EB·EA = EC·ED, a given magnitude; theref. G is a given point, and a circle will paſs through the points G, F, C and D; therefore the ∠ HCG = FGC = FDC = CHG, wheref. HG = CG; Draw KI perpendicular to HC, and GI perpendicular to HC alſo, and HI = IC, theref. GK is one continued ſtraight line, that is, the line GL, drawn through the centre K, will be perpendicular to HC or EF, and therefore the point O becomes known by *Lawſon's* 7th Propoſition: hence the poſition of the diagonal AC is given.

CONSTRUCTION. From the centre K draw KG perpendicular to EF, draw any line GH to meet the circle in A and H, and HC perpendicular to KG to meet the circle in C and join A, C. Through the point of interſection of AC, GK, and the given point draw the diagonal AC, compleat the trapezium, which will be inſcribed in the given circle when the points are ſo ſituated as

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as to render the problem poſſible*. For the $\angle$ EGA = FGC, and EAG = GFC; therefore EG : GA :: GC : GF or EG GF = GC GA = GL GT; and GL perpendicular to EF, therefore the points A, C, D, and B, are in the circle by Prop. 33, 34, I. *Stewart's Propoſitiones Geometricæ.*

The ſame, answered by Mr. James Cunliffe.

Fig. 535, Pl. 28. ANALYSIS. Let ABCD be a trapezium inſcribed in a circle, the ſides BA, CD being produced to meet in the point E; and the ſides AD, BC being produced to meet in the point F. Draw the tangents ET, FM, and to the centre O draw EO, FO, TO, MO; alſo draw Dn to meet FE in n; making the $\angle$ EnD = DAB, and join On.

Then the $\angle$ EnD + DAE = DAB + DAE = 2 right angles; whence the points E, A, D, n, lie in the circumference of the ſame circle, per Euc. 22, III. Again, ſince, by conſtruction, the $\angle$ EnD = DAB, the ſupplements of theſe angles muſt be equal, viz. $\angle$ FnD = DCB, conſ. q. the points F, C, D, n, lie in the circumference of the ſame circle;

wherefore, per Euc. 36, III, $Fn \times FE = FD \times FA = FM^2$,

and $En \times FE = ED \times EC = ET^2$;

and, by addition, $Fn \times FE + En \times FE = FE^2 = FM^2 + ET^2$. From whence it appears that the points E and F, muſt always have ſuch a ſituation with reſpect to the circle, that the ſquare of the diſtance between them may be equal to the ſum of the ſquares of two tangents from them to the circle.

The $\angle$s ETO, and FMO are right $\angle$s, per Euc. 18, III; therefore $FO^2 - OM^2 = FO^2 - OT^2 = FM^2 = Fn \times FE$,

and $EO^2 - OT^2 = ET^2 = En \times FE$

and theref. $FO^2 - EO^2 = Fn \times FE - En \times FE = FE \times (Fn - En)$. Therefore O$n$ is perpendicular to FE, as appears from prop. 24, B. II, *Emerſon's Geometry.*

Draw MR $\perp$ to FO and join ER; the $\triangle$s FRM and FMO are right angled and ſimilar, theref. FR : FM :: FM : FO, whence $FR \times FO = FM^2$; and it has been ſhewn that $Fn \times FE = FM^2$; conſeq. $FR \times FO = Fn \times FE$, wheref. per Euc. 36, III, the points E, n, R, O, are in the circumference of a circle, and theref. per Euc. 21, III, $\angle$ ERO = FnO = a right angle; wheref. ERM is a ſtraight line and cuts FO at right angles in R. And in the ſame manner it may be ſhewn that FT cuts EO at right angles in Q; and theref. it is plain from prop. 34, B. II. *Emer-*

* *Note.* The points muſt be ſituated as mentioned in Mr. Swale's ſolution.

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fon's Geom. that the lines FT, EM and On interfect in the fame point I.

Thefe things being premifed the *conftruction* of the problem may be as follows.

Let E and F be the two given points in which the oppofite fides of the trapezium are to interfect, fo fituated with refpect to the given circle TmtM, as to render the queftion poffible; alfo let P be the given point through which one of the diagonals of the trapezium is to pafs. Having found the point I from what is fuggefted in the *Analyfis*, through that and the given point P draw the right line DB cutting the circle in D and B; alfo draw EB cutting the circle again in A, and through I draw AIC to cut the circle in C, then draw the lines BC, CD, DA, and ABCD is the required trapezium.

DEMONSTRATION. Draw the lines AQ, BQ, AO, BO, ED, FD and FC; alfo draw OS $\perp$ to AB. Then from what is deduced in the Analyfis, and Euc. 36, III. $ET^2 = EA \times EB = EQ \times EO$, wherefore the points A, B, O, Q, lie in the circumference of the fame circle; and per Euc. 20, 21, 22, III, $\angle BQA = BOA = 2BDA$ (2IDA), and $\angle AQE + AQO = ABO + AQO = 2$ right angles, whence $\angle AQE = ABO$: let each of thefe be taken from a right angle and there will remain $\angle AQT = BOS = BCA$ (BCI); and confequently $\angle BQT = AQT = BCA = BDA$ (IDA). Again from what is deduced in the *Analyfis* and Euc. 36, III, $FM^2 = FR \times FO = FI \times FQ = FI \times (FI + IQ) = FI^2 + FI \times IQ$: and per prop. 27, B. II. *Emerfon's Geom.* $FM^2 = FI^2 + mI \times IM$, whence, and per Euc. 35, III, $FI \times IQ = mI \times IM = AI \times IC$, wherefore the points A, Q, C, F, lie in the circumference of the fame circle, and per Euc. 21, III, $\angle AQI = FCI$, whence, by addition, $\angle AQT + AQI = BCI + FCI = 2$ right angles; therefore FCB is a right line. Again fince $FI \times IQ = AI \times IC = DI \times IB$, the points F, D, Q, B, lie in the circumference of the fame circle, and $\angle BQI = IDF$, whence, by addition, $\angle BQT + BQI = IDA + IDF = 2$ right angles; therefore FDA is a right line. Now having demonftrated that FCB and FDA are right lines:—draw the lines BR, CR, and by proceeding as before, it may be fhewn that $\angle BRM = CRM = IDC$, and $\angle BRI = IDE$, whence, by addition, $\angle BRM + BRI = IDC + IDE = 2$ right angles; therefore EDC is a right line; and therefore ABCD is the required trapezium.

The following particulars naturally fuggeft themfelves, from what is deduced in the *Analyfis*, namely,

A line joining the points of contact of two tangents to the circle from either of the given points being produced will pafs through the other given point.

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The angle EOF must be less than a right angle.

For, $OT^2 = OQ \times OE = OR \times OF$; but when EOF is a right angle the points Q, R and O, coincide, and therefore the radius OT becomes $= 0$, or the circle degenerates to a point.

Three points being given in any position whatever, so that right lines joining them may form an acute angled triangle, a circle may be found having its centre in one of them, such that the opposite sides of a trapezium, inscribed therein, being produced will intersect in the other two, and have one of its diagonals parallel to a right line given by position.

If we imagine the shadow of the figure to be projected from a lucid point upon a plane inclined to the plane of the figure, the circle will be projected into a conic section with an inscribed trapezium, whose opposite sides produced intersect in two points so situated with respect to the conic section, that a right line joining the points of contact of two tangents to the curve from either of the points of intersection, will pass through the other. This may be gathered from what is deduced in the *Analysis*, Cor. 1. Prop. 20, B. II. *of Emerson's Nature and Property of curve lines.*

Whence it follows, that if instead of a circle, the curve had been a given conic section, the method of construction would have been in no respect different:—but then the given points must have had such a situation, that the right line joining the points of contact of two tangents from either of them would pass through the other.

The same, answered by the proposer, Mr. Lowry.

Analysis. Fig. 537, Pl. 28. Suppose the thing done, ABCD the trapezium required; P, Q, the given points where the opposite sides produced intersect; and S the point through which the diagonal passes. Let O be the centre of the given circle, and draw the tangents PE, QF. Join O, E; O, F; O, P; O, Q; and draw PF to meet the circle at H, and OQ at L; and QE to meet OP at K, PF at I, and the circle at G. Join also, P, Q; P, G; and Q, H.

Then because the question requires the trapezium to be inscribed in the circle, it is evident from *Lawson's* 33rd proposition, that the position of the points P and Q, with respect to the circle, must be such that $PQ^2 = PE^2 + QF^2$. Now QF^2 is $= QO^2 - OF^2 = QO^2 - OE^2$; theref. $PQ^2 - PE^2 = QO^2 - OE^2$, and conseq. PO is perpendicular to EQ. In like manner it may be shewn that QO is perpendicular to PF; therefore PG is $=$ PE, and QH $=$ QF: Wherefore *Ibid.* Prop. 18. PE or HPF $= RI^2 + EIG = PI^2 +$ HIF,

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HIF, and QF² or GQE = QI² + GIE. Whence it follows from Prop. 34, *Lib*, that if any ſtraight line PAB be drawn meeting the circle at A and B, and AIC, PDC be drawn meeting the circle at C and D, the points D, I, B, will be in a ſtraight line; and if any line QDA be drawn meeting the circle in A and D, and DIB, QCB be drawn meeting the circle in B and C, the points A, I, C, will be in a ſtraight line; and becauſe the points P and Q are ſo poſited that a ſtraight line will paſs through the points A, D, Q, and alſo through the points B, C, Q, it neceſſarily follows, that if any trapezium whatever whoſe oppoſite ſides, when produced, meet in P and Q, be inſcribed in the circle, the diagonals AC, BD, will always interſect each other in the point I(*).

Now becauſe I and S are given points, the diagonal DB will be given by poſition, and the *Conſtruction* may be as follows. Draw the tangents PE, QF, and let PF, QE be drawn interſecting at I; through I and S draw the diagonal DISB, join PAB, PDC, QDA and QCB, and it is done, as is manifeſt from the Analyſis.

If only one of the points of interſection had been given, and the other had been required to fall in a line *of any order* given by poſition, the conſtruction would have been very little different from the above. For it is obvious, that the ſtraight lines FIP, EIQ, are the *loci* of the points P, Q.

That the diagonals always interſect each other in the given point I, might be proved differently as follows... For, it is manifeſt from Lawſon's Propoſitions already quoted, that if from P, any two ſtraight lines whatever, as PAB, PDC, be drawn meeting the circle in A, B; D, C; the *locus* of the point of interſection of AC, BC, will be the ſtraight line EG; and, ſuppoſing, the point P, and the lines PAB, PDC, to be removed, and from Q any two ſtraight lines whatever, as QDA, QCB, be drawn meeting the circle at A, D; CB; the *locus* of the point of interſection interſection of AC, BD, will be the ſtraight line HF. Now, when the points P, Q, are ſo ſituated that a trapezium can be inſcribed in the circle, it is obvious, that the point of interſection of AC, BD, muſt be a point which is common to each of the *loci* EG, FH, and conſequently it muſt be the point I, where they interſect.

It was from this conſideration that I was firſt led to the diſcovery of the beautiful PORISM exhibited in the above Analyſis, (*), and at that time I was not aware that it had been thought of by any one. Since then a literary friend has favoured me with a copy of the *Opera reliqua* of Dr. ROBERT SIMSON, where I find a compleat inveſtigation is given of this Poriſm.

To ſuch of the readers of the Repoſitory as have not had an opportunity of ſeeing that valuable work, an abſtract of the DR.'s

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inveſtigation of this *Poriſm* may perhaps be acceptable. Previous to the inveſtigation he gives the two following propoſitions.

PROP. I. *Being Prop. LIX. De Porismatibus.*

Fig. 538, Pl. 28. If from two points A, B, without a circle, two ſtraight lines AC, BC, be inflected to the circumference, meeting it again in D and E, and the ſum of the rectangles CA, AD; CE, BE, be equal to the ſquare of AB joined; and M be the centre of the circle, and MK be drawn perpendicular to AB, then $BA \cdot AK = CA \cdot AD$.

And the contrary. If the ſum of the rectangles CA, AD; CB, BE; be $= AB^2$, and in AB there be taken a point K, ſuch, that $BA \cdot AK = CA \cdot AD$, then if KM be drawn to the centre it will be perpendicular to AB.

Let KM meet the circumference in N, O; then $CA \cdot AD = AK^2 + OK \cdot KN$, and in like manner $EB \cdot BC = BK^2 + OK \cdot KN$; therefore $CA \cdot AD + CB \cdot BE$ (that is, by hypotheſis AB^2) $= AK^2 + BK^2 + 2OK \cdot KN$; take away $AK^2 + BK^2$, and $2AK \cdot KB = 2OK \cdot KN$; wherefore $AK \cdot KB = OK \cdot KN$; add AK^2 to each, and $BA \cdot AK = OK \cdot KN + AK^2$, that is $= CA \cdot AD$. In the ſame way it may be ſhewn that $AB \cdot BK = CB \cdot BE$.

And if the rectangle $CA \cdot AD + CB \cdot BE$ be $= AB^2$, and in AB the point K be taken making $BA \cdot AK = CA \cdot AD$; then if KM be drawn to the centre, it will be perpendicular to AB.

For, if it is not, from the centre draw MR perpendicular to AB; then by the preceding $BA \cdot AR = CA \cdot AD$, that is $= BA \cdot AK$, which is impoſſible; therefore MK is perpendicular to AB.

PROP. II. *Being Prop. LX. De Poriſmatibus.*

Fig. 539, Pl. 28. If from two points A, B, of which A is without, and B within the circle, two ſtraight lines AC, BC, be inflected to the circumference, meeting it again in D and E; and the rectangle $CA \cdot AD$ be equal to the rectangle CB BE, together with the ſquare of AB joined; and from the centre MK be drawn perpendicular to AB, then the rectangle $BA \cdot AK = CA \cdot AD$.

And the contrary, if $CA \cdot AD$ be $= CB \cdot BE + AB^2$, and in AB there be taken the point K, ſuch, that $BA \cdot AK = CA \cdot AD$, then if KM be drawn to the centre it will be perpendicular to AB.

Let AB meet the circumference in S, T; and becauſe the rectangle $CA \cdot AD$, that is, $TA \cdot AS$ is $= CB \cdot BE$ or $SB \cdot BT + AB^2$, let SK^2 be added to each, then (6. II) $AK^2 = SB \cdot BT + AB^2 + SK^2$; take away AB^2, then $2AB \cdot BK + BK^2 = SB \cdot BT + SK^2$. Again, take away, BK^2, and $2AB \cdot BK$ is $= 2SB \cdot BT$; therefore

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AB·BK = SB·BT. Add AB² to each, and BA·AK = SB·BT + AB² or CB·BE + AB², that is, by hypothesis, = CA·AD.

The second part of the proposition is proved after the same manner as the second part of the preceding.

Now, the PORISM, as enunciated by Dr. SIMSON, is as follows. Let A, B, be two given points, and to the circumference of a circle CDE given by position, let any two lines AC, BC, be inflected meeting it again in D and E, the straight line which joins the points D, E, will pass through a given point.

CASE I. When the straight line which joins the given points A, B, does not meet the circle. Fig. 538.

Let F be the point sought, and let BD be drawn meeting the circle again in G; then because from the points A, B, to the point D, two straight lines AD, BD, are inflected, and which meet the circumference again in C and G; the straight line which joins CG, will, by hypothesis, pass through the same point F. Again, draw AG, and because from the points A, B, to G, the straight lines AG, BG, are inflected one of which, as BG, meets the circumference again in D, the straight line which joins the point D, and the point of intersection of the remaining line AG and the circle, will also, by hypothesis, pass through the point F; therefore DF, when drawn, will pass through the point where AG intersects the circle: but DF meets the circumference in E. Wherefore the point E is the intersection of AG and the circumference, that is the points A, G, E, are in a straight line.

Let EH, drawn parallel to AB, meet the circumference again in H, and let DH joined meet AB in K. Then because from the points A, B, the straight lines AC, BC, are inflected to the circumference meeting it again in D and E, and EH is parallel to AB, and DH meets AB at K, the rectangle BA·AK = CA·AD*, therefore B, K, D, C are in a circle, and consequently the rectangle AKD = DCE = AGD (22. III.). therefore the points A, K, G, D, are in a circle; consequently AB·BK = DB·BC = CB·BE. Therefore because it has been shewn that BA·AK = CA·AD, and AB·BK = CB·BE; CA·AD + CB·BE will be = BA·AK + AB·BK, that is, = AB². Let M be the centre of the circle, and let MK, when joined, meet the circumference in N and O, then MK is perpendicular to AB, and consequently to HF (Prop. I.). Therefore, because HE is perpendicular to the diameter NO, and HD is drawn to the circumference and meets the diameter in K, the straight line ED passes through a given point in the diameter, viz. that which divides it in the same

* *Prop. 55. De Porismatibus. The Demonstration is evident.*

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ratio as NK to KO*. But DE, by hypothesis, passes through a given point F, therefore F is in the diameter: for if not, there would be two points given in the straight line DE, and consequently it would be given by position; but this is contrary to the hypothesis. Therefore DE passes through a given point, when $AB^2 = CA \cdot AD + CB \cdot BE$, and the point F is found by making NK : KO :: NF : FO.

COMPOSITION. Let there be two given points A, B, and to any point in the circumference of the circle CDE given by position, let AC, BC, be inflected meeting it again in D and E, and let $AB^2 = CA \cdot AD + CB \cdot BE$; then DE, when joined, will pass through a given point F, which is found by drawing MK through the centre M perpendicular to AB, and making NK : KO :: NF : FO.

For, let KD be joined meeting the circle again in H, and let DE meet the diameter in F, and join HE: then because $CA \cdot AD + CB \cdot BE = AB^2$, and MK is drawn perpendicular to AB, $BA \cdot AK = CA \cdot AD$; therefore HE is parallel to AB, and consequently perpendicular to diameter NMO. And because HD meets the diameter in K, and DE meets it in F, NK : KO :: NF : FO: but NK and KO are given, therefore the point F is given through which DE passes. *Q. E. D.*

CASE II. When the straight line which joins the given points A, B, meets the circle. Fig. 539.

It may be shewn as in *Case* 1, that, $BA \cdot AK = CA \cdot AD$, and $AB \cdot BK = CB \cdot BE$; therefore $CA \cdot AD = (BA \cdot AK$ or $AB \cdot BK + AB =) CB \cdot BE + AB^2$. and if MK be drawn to the centre it will be perpendicular to AB (Prop. II.), and consequently to HE; therefore because HE is perpendicular to the diameter NO, and HD is drawn to the circumference and meets the diameter in K, the straight line DE passes through a given point in the diameter produced †; but, by hypothesis, DE passes through a given point F, therefore F is in the diameter produced. For, if it is not, two points will be given in the straight line DE, and consequently it would be given by position, but this is contrary to the hypothesis; therefore DE passes through a given point when $CA \cdot AD = CB \cdot BE + AB^2$, and the point F is found by making NK : KO :: NF : FO.

DR. SIMSON then gives the *Composition*, but as it differs so little from that of the first case we shall omit it.

* *Prop.* 51, *De Porismatibus, or Prop.* 156, *Lib.* 7. *Pap. or Prop.* 7, *Lawson.*

† *Lawson's 7th Proposition.*

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Ingenious ſolutions were alſo received from Meſſrs. Hill, Rutherford, Thornoby, *and* Whalley.

Mr. Swale *is requeſted to ſend to Mr.* Glendinning's *for the Medal for ſolving the Prize Queſtion.*

ARTICLE V.

MATHEMATICAL QUESTIONS.

To be anſwered in Number XIV.

I. QUESTION 271, *by* Merones Minor.

SUPPOSE a man has a calf which at the end of three years begins to breed, and afterwards brings a female calf every year; and that each calf begins to breed in like manner at the end of three years, bringing forth a cow calf every year; and that theſe laſt breed in the ſame manner &c.; to determine the owners whole ſtock at the end of 30 years.

II. QUESTION 272, *by Mr.* W. Marrat, *Boſton.*

A tub full of water, in form of a conical fruſtum, the inner dimenſions of which are as follow, viz. the top and bottom diameters 20 and 30, and the perpendicular altitude 25 inches, being ſet upon an inclined plane, 2 gallons ale meaſure were obſerved to run over. What was the plane's inclination to the horizon?

III. QUESTION 273. *by Mr.* Olinthus Gregory, *Cambridge.*

The curve ſurface of a cone is 753·98232 inches, and its ſolidity is 2412·743424 inches: it is required to find at what diſtance from the vertex of the cone (meaſuring on the ſlant ſide) an ungula muſt be cut off by a vertical plane, ſo as to take away a fifth of the curve ſurface?

IV.

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IV. QUESTION 274, *by Miſs* Suſan May.

Two given weights as y and z,
Connected by a ſilken thread,
Suſpended o'er a pulley are,
Whoſe radius we'll denote by r;
A motion will of courſe enſue,
(The great will ſtill the ſmall ſubdue);
The preſſure on the pulley's axe,
Ye learned youths is what I aſk?

V. QUESTION 275, *by* Merones Minor.

Fig. 536, pl. 28, AEB is a ſemicircle, BD is perpendicular and equal to AB, the diameter; DFM is any right line drawn from D, cutting the circumference in F and the diameter in M, and MP is perpendicular to AB and equal to MF. Required the quadrature of the curve which is the locus of P?

VI. QUESTION 276, *by Mr.* W. Peacock, *Birmingham.*

A ſpherical ball of copper fixed to the end of a ſlender rod whoſe length was l inches, and weight w ounces, was obſerved to make v vibrations in s ſeconds: Quere the weight of the ball?

VII. QUESTION 277, *by the Rev.* David Davies, *Swanſea.*

A cylinder is juſt immerſed in a fluid with its axis perpendicular to the ſurface; find at what point it muſt be cut by a plane perpendicular to its axis, that the preſſures upon the convex ſurfaces may be equal?

VIII. QUESTION 278, *by the Rev.* D. Davies.

A body falls freely by the force of gravity down AB, and uniformly deſcribes the ſpace BD equal to twice AB on the horizontal plane with the velocity acquired: Determine geometrically the length and inclination of a plane drawn from A to BD, the time down which may be equal to the time down AB and along BD.

IX.

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IX. QUESTION 279, *by* Amicus.

Two equal bodies A and B move at the ſame time from a point in the periphery of a given circle the one along the periphery and the other along a chord given by poſition.—They move in ſuch a manner that the ſtraight line connecting them always paſſes through the centre of the circle: it is required to determine the nature of the track deſcribed by the centre of gravity of the two bodies?

X. QUESTION 280, *by Miſs* May.

Required the area of the greateſt rectangle that can be inſcribed in the harmonic curve whoſe baſe is 20 and height 2?

XI. QUESTION 281, *by Mr.* John Dawes, *Birmingham.*

Given the ſum of the diſtances between the centres of the ſun (S), moon (L), and a fixt ſtar (A) = 183° 18′; the ∠ ASL = 60°, and the ∠ ALS = 107° 20′, to find each diſtance.

XII. QUESTION 282, *by Mr.* George Roy.

It is, I believe, pretty generally underſtood that the decimal obtained by extracting any root of a whole number cannot be a repetend. An inveſtigation of this is required.

XIII. QUESTION 283, *by Mr.* John Andrew, *Cark.*

Given the difference between the ſegments of the baſe made by the point of contact with the inſcribed circle and that made by the perpendicular, the difference of the ſides and the line biſecting the vertical angle, to conſtruct the plane triangle.

XIV. QUESTION 284, *by Mr.* W. Paſſman, *Hull.*

I demand the diameter of the greateſt circle that can be inſcribed in the quadrant of a given ellipſis?

XV.

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XV. QUESTION 285, *by Mr.* Nicholas Bottom.

Given the baſe, the verticle angle, and aggregate of the ſum of the ſides and line biſecting the vertical angle, to conſtruct the plane triangle.

XVI. QUESTION 286, *by Mr.* Gregory.

In fig. 534, pl. 28, BC is a balance, at one end of which hangs a ſcale CD in which ſits a man, who is counterpoiſed by the weight W at the end B; BF and FC being equal. Now if the man puſhes a ſtaff againſt the point P (FP being $\frac{1}{7}$ of FC) with a force of 8 ſtone in the direction DP making the angle DPC = 60°; and the poſt GH prevent the ſcale CD from leaving its perpendicular poſition; it is required to find what weight muſt be added to, or taken from, W, ſo that the equilibrium may be maintained, while the preſſure on P in the direction DP is exerted?

XVII. QUESTION 287, *by Mr.* Samuel Thornoby.

The ſame things being ſuppoſed as in queſtion 287; it is required to determine when the tradeſman's debt will be the greateſt poſſible?

XVIII. QUESTION 288, *by* Merones Minor.

Given $l = 30$, the length; $b = 24$, the breadth; and $h = 4$, the heights of a groin formed by circular arches; to find the ſuperficies and vacuity by a clear and general inveſtigation?

XIX. QUESTION 289, *by Mr.* Johnſon, *Birmingham.*

To find a point ſuch that if perpendiculars be drawn from it to the ſides of a given trapezium, the ſum and ſum of the ſquares of thoſe perpendiculars may be given magnitudes.

XX. QUESTION 290, *by Mr.* Gregory.

For a certain period during every year there is no real night at Cambridge in latitude 52° 12′ 35″ N: it is required to ſhew the time in the morning when the ſun's azimuth from the north is a maximum, on the 1ſt day of this period in a place whoſe latitude north is half the complement of the latitude of Cambridge.

XXI.

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XXI. QUESTION 291, *by Mr.* Louis Hill.

It is required to divide a right angle into three others such that their ſines may obtain a given ratio?

XXII. QUESTION 292, *by Mr.* Nicholas Bottom.

If a heavy flexible line or chain, of a given length, faſtened together at the ends, be ſuſpended by a tack, at a given point upon the perfectly poliſhed ſurface of a given right cone in ſuch a manner as to include the vertex thereof: it is required to determine the portion of the convex ſurface included by the chain.

XXIII. QUESTION 293, *by Mr.* James Cunliffe, *Bolton.*

Required the ſum of n terms of the ſeries,

$$1 \cdot 2x + 2 \cdot 3x^2 + 3 \cdot 4x^3 + 4 \cdot 5x^4 + 5 \cdot 6x^5 + \&c.\ ?$$

XXIV. QUESTION 294, *by Mr.* Lowry, *Birmingham.*

Given the ſides of a trapezium to conſtruct it when one of the diagonals divides the other in a given ratio.

XXV. QUESTION 295, *by Mr.* Johnſon, *Birmingham.*

Let there be any number of ſtraight lines A*a*, B*b*, C*c*, D*d*, &c. given by poſition and parallel to each other in which are the given points A, B, C, D, &c. It is required to draw a ſtraight line to paſs through a given point P, and interſect the parallels at L, M, N, O, &c. ſo that the ſum of the ſquares of AL, BM, CN, DO, &c. may be equal to a given ſpace?

XXVI. QUESTION 296, *by Mr.* James Cunliffe.

To find the leaſt poſitive values of x and y in whole numbers, ſo that $x^2 + xy + y^2$ and $x^2 + y^2$ may be both rational ſquares?

XXVII. QUESTION 297, *by Mr.* John Lowry.

AB and AC are ſtraight lines given by poſition and P and Q are given points. It is required to draw a ſtraight line through Q to

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to meet AB, AC, in B and C, so that if BP, CP be joined, the triangle BPC may be of a given magnitude; and to shew the limits?

XXVIII. QUESTION 298, *by Mr.* W. Smith, *Liverpool.*

From a given point P, in AB, one of the sides, produced, of a given triangle ABC, it is required to draw a straight line PDE meeting the other two sides AC, BC, in D and E, so that if DE be drawn parallel to AB, the side in which is the given point, meeting BC in F, the triangle DEF may be a maximum?

XXIX. QUESTION 299, *by Mr.* Lowry.

From two given points A and B, situated in the circumference of a circle given in magnitude and position, it is required to draw two straight lines AEG, BEH to intersect the circle at E and meet two straight lines XG, YH, given by position, in which are the given points X, Y, at G and H, so that the sum or difference of the squares of XG, YH may be of a given magnitude.

XXX. PRIZE QUESTION 300, *by* Yanto.

If Q be the point in a triangle from which perpendiculars are drawn to the sides of the triangle, so that the sum of their squares is the least possible; twice the area of the triangle is a mean proportional between the sum of the squares of the sides of the triangle, and the sum of the squares of the above-mentioned perpendiculars. Required a Demonstration?

ARTICLE VI.

SIX PROPOSITIONS FROM LAWSON.

(To be answered in Number XIV.)

PROP. LV.

IN every right-angled triangle, as the difference between the hypothenuse and one side is to the difference between the same side and its adjacent segment, so is the same side to the same segment.

PROP.

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PROP. LVI.

If HC be a tangent to a circle meeting the diameter DB produced in H, and from the point of contact C a perpendicular CK to that diameter be drawn, and likewise a line from H cutting the circle in F and G, and the perpendicular CK in I, and F be the nearest point to H; then I say that the square of HF is to the square of the tangent HC as FI to IG.

PROP. LVII.

If one side AC of an equilateral triangle ABC be produced to E so that CE may be equal to AC, and from A a perpendicular to AC raised, and from E a line drawn through the vertex B to meet the perpendicular in D; then I say that BD is equal to the radius of the circle which circumscribes the triangle.

PROP. LVIII.

If BD bisect the vertical angle B of a triangle ABC and meet the base in D, and if with either of the other angular points A or C as centre and the adjacent segment of the base as radius a circle be described to cut BD again in E; then I say that BE is to BD as that segment used as radius is to the other.

PROP. LIX.

If BD bisect the vertical angle B of the triangle ABC, and if on BA or BC from B be put a third proportional to the other side and the bisecting line; then I say the rectangle under that side on which it is put and its remainder when the third proportional is taken from it is equal to the square of the adjacent segment of the base made by the bisecting line, that is, BCE = CD^2, or BAE = AD^2.

PROP. LX.

If an isosceles triangle be inscribed in a semi-circle and one of the equal sides produced, and if from any point E in the diameter a perpendicular thereto be drawn to cut the side, the circle, and the side produced in the points G, H, and F respectively; then I say that EG, EH, and EF are continual proportionals.

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ARTICLE VII.

To the Editor of the Repository.

SIR,

IF you should look upon the following attempt to investigate the Theorem, given by Dr. Waring, for finding the sum of *any* power, of the roots of an equation, as deserving of a place in your valuable Repository, I should be glad if you will insert it; but if you know of any paper upon the same subject which will supercede the use of this, I must beg the favour that it may not be printed. The writer of this article having but a very confined acquaintance with Mathematical Books, there may be several demonstrations of this Rule which he has not seen.

I am, Sir, your humble servant,

A. B.

Of the Sums of the Powers of the Roots of an Equation.

LET $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \&c. = 0$,

be the given equation, whose roots are a, b, c, d, &c.

It is well known that $\frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c} + \frac{1}{x-d} + \&c.$ is $=$

$$\frac{nx^{n-1} - (n-1)\,px^{n-2} + (n-2)\,qx^{n-3} - (n-3)\,rx^{n-4} + \&c.}{x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \&c.} =$$

suppose to $M \div N$; therefore,

$$\frac{1}{x-a} - 1 + \frac{1}{x-b} - 1 + \frac{1}{x-c} - 1 + \frac{1}{x-d} - 1 = \frac{M}{N} - n,$$

the number of factors being n; hence

$$\frac{1-x+a}{x-a} + \frac{1-x+b}{x-b} + \frac{1-x+c}{x-c} + \&c. = \frac{M - nN}{N}$$

Multiply the first equation by $x-1$, then

$$\frac{x-1}{x-a} + \frac{x-1}{b} + \frac{x-1}{x-c} + \&c. = \frac{xM-M}{N},$$

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add two equations together, and

$$\frac{a}{x-a}+\frac{b}{x-b}+\frac{c}{x-c}+\frac{d}{x-d}+\&c.=\frac{x\mathrm{M}-n\mathrm{N}}{\mathrm{N}}$$

$$\frac{px^{n-1}-2qx^{n-2}+3rx^{n-3}-4sx^{n-4}+\&c.}{x^n-px^{n-1}+qx^{n-2}-rx^{n-3}+sx^{n-4}+\&c.},$$

wherefore, by actual diviſion,

$$\left.\begin{array}{l}\frac{a}{x}+\frac{a^2}{x^2}+\frac{a^3}{x^3}+\&c.\\ +\frac{b}{x}+\frac{b^2}{x^2}+\frac{b^3}{x^3}+\&c.\\ +\frac{c}{x}+\frac{c^2}{x^2}+\frac{c^3}{x^3}+\&c.\end{array}\right\}=\left\{\begin{array}{l}\frac{p}{x}+\frac{p^2-2q}{x^2}+\frac{p^3-3qp+3r}{a^3}\\ +\frac{p^4-4qp^2+4rp+2q^2}{x^4}+\frac{+4s}{}+\&c.\end{array}\right.$$

The homologous terms, or thoſe where x is raiſed to the ſame power muſt be equal to each other, therefore

$a+b+c+\&c.=p$, $a^2+b^2+c^2+\&c.=p^2-2q$.

$a^3+b^3+c^3+\&c.=p^3-3qp+3r$,

$a^4+b^4+c^4+\&c.=p^4+4qp^2+4rp+2q^2-4s$ &c;

for it is evident that the 1ſt, 2nd, 3rd, &c. numerators in the quotient, are reſpectively equal to the ſum of the 1ſt, 2nd, 3rd, &c. powers of the roots of the given equation; therefore if the ſum of the mth powers of the roots be required, continue the diviſion till the quotient conſiſts of m terms, and the numerator of the laſt term is the ſum required.

It is evident that the *numerator* of the fraction

$$\frac{px^{n-1}-2qx^{n-2}+3rx^{n-3}-4sx^{n-4}+\&c.}{x^n-px^{n-1}+qx^{n-2}+rx^{n-3}+\&c.},$$ is found by multiplying the terms of the given equation reſpectively by the terms of the following ſeries, viz. 0, — 1, — 2, — 3, — &c.

Becauſe the co-efficients p, q, r, s, &c. are quantities of one, two, three, &c. dimenſions, if therefore the ſum of the mth powers of the roots be wanted in any equation of n dimenſions (n being greater than m) the numerator of the mth term in the quotient given by reducing, or dividing the following fraction, viz.

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$$\frac{px^{n-1} - 2qx^{n-2} + 3rx^{n-3} \ldots\ldots \mp mwx^{k-m}}{x^n - px^{n-1} + qx^{n-2} \ldots\ldots \pm wx^{n-m}} =$$

$\frac{px^{m-1} - 2qx^{m-2} + 3r^{m-3} \ldots\ldots \mp mw}{x^m - px^{m-1} + qx^{m-2} \ldots\ldots \pm w}$, will be the sum required.

From the above it follows as an evident corollary that because

$$\frac{px^{n-1} - 2qx^{n-2} + 3rx^{n-3} - 4sx^{n-4} + \&c.}{x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c.} =$$

$$\left.\begin{array}{l} \frac{a^0}{x^0} + \frac{a^1}{x^1} + \frac{a^2}{x^2} + \frac{a^3}{x^3} + \&c. \\ + \frac{b^0}{x^0} + \frac{b^1}{x^1} + \frac{b^2}{x^2} + \frac{b^3}{x^3} + \&c. \\ + \frac{c^0}{x^0} + \frac{c^1}{x^1} + \frac{c^2}{x^2} + \frac{c^3}{x^3} + \&c. \\ \&c. \quad \&c. \quad \&c. \quad \&c. \end{array}\right\} \text{in infinitum,}$$

therefore, if $x = 1$,

$$\frac{p - 2q + 3r - 4s + \&c.}{1 - p + q - r + s - \&c.} = \left\{\begin{array}{l} a + b + c + \&c. \\ + a^2 + b^2 + c^2 + \&c. \\ + a^3 + b^3 + c^3 + \&c. \\ \&c. \text{ in infinitum} \end{array}\right.$$

that is, the sum, of the sum of the roots, the sum of the squares, cubes, &c. in infinitum of an equation of n dimensions is equal to $\frac{p - 2q + 3r - 4s + \&c.}{1 - p + q - r + s - \&c.}$, if that sum be finite.

To find the sum of the mth powers of their roots:

Put $R = \frac{p}{x^1} - \frac{q}{x^2} + \frac{r}{x^3} - \frac{s}{x^4} + \&c.$ then because

$\frac{1}{1-R} = 1 + R^1 + R^2 + R^3 + \&c.$ therefore.

$$\frac{px^{n-1} - 2q^{xn-2} + 3rx^{n-3} - 4sx^{n-4} + \&c.}{x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} \&c.} =$$

$$\left(\frac{p}{x} - \frac{2q}{x^2} + \frac{3r}{x^3} - \&c.\right) \div \left(1 - \frac{p}{x} + \frac{q}{x^2} - \frac{r}{x^3} + \&c.\right) =$$

$$\left(\frac{p}{x} - \frac{2q}{x^2} + \frac{3r}{x^3} - \&c.\right) \times \left(1 + R + R^2 \ldots R^{m-2} + R^{m-1} \&c.\right) =$$

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$$\left(\frac{p}{x}-\frac{2q}{x^2}+\frac{3r}{x^3}-\&c.\right)\times\left(R^{m-1}+R^{m-2}+R^{m-3}\ \ldots..R+1\right).$$

Hence by ſubſtituting the value of R, we ſhall have,

$$\left.\begin{array}{l}\frac{a}{x}+\frac{a^2}{x^2}+\frac{a^3}{x^3}\ \ldots.+\frac{a^m}{x^m}\ \&c.\\ +\frac{b}{x}+\frac{b^2}{x^2}+\frac{b^3}{x^3}\ \ldots.+\frac{b^m}{x^m}\ \&c.\\ +\frac{c}{x}+\frac{c^2}{x^2}+\frac{c^3}{x^3}\ \ldots.+\frac{c^m}{x^m}\ \&c.\\ \&c.\ \ \&c.\ \ \&c.\ \ \ \&c.\end{array}\right\}=\left(\frac{p}{x}-\frac{2q}{x^2}+\frac{3r}{x^3}-\&c.\right)\times$$

$$\left\{\begin{array}{l}\frac{1}{x^{m-1}}\left(p-\frac{q}{x}+\frac{r}{x^2}-\frac{s}{x^3}+\frac{t}{x^4}-\&c.\right)^{m-1}\\ +\frac{1}{x^{m-2}}\left(p-\frac{q}{x}+\frac{r}{x^2}-\frac{s}{x^3}+\frac{t}{x^4}-\&c.\right)^{m-2}\\ +\frac{1}{x^{m-3}}\left(p-\frac{q}{x}+\frac{r}{x^2}-\frac{s}{x^3}+\frac{t}{x^4}-\&c.\right)^{m-3}\\ \ldots..\\ +\frac{1}{x^3}\left(p-\frac{q}{x}+\frac{r}{x^2}-\frac{s}{x^3}+\frac{t}{x^4}-\&c.\right)^3\\ +\frac{1}{x^2}\left(p-\frac{q}{x}+\frac{r}{x^2}-\frac{s}{x^3}+\frac{t}{x^4}-\&c.\right)^2\\ +\frac{1}{x}\left(p-\frac{q}{x}+\frac{r}{x^2}-\frac{s}{x^3}+\frac{t}{x^4}-\&c.\right)\\ +1\end{array}\right\}$$

It is evident from the mere inſpection of the laſt equation, that the firſt term of the higheſt line, the ſecond of the next, the third of the next following, &c. are homologous, or contain the ſame power of x, viz. $\frac{1}{x^{m-1}}$; or more generally, the firſt term of any line, the ſecond of the line below, the third in the next following, &c. are homologous. If therefore we put

A, A′, A″, &c. for the 1st terms of R^{m-1}, R^{m-2}, R^{m-3}, &c. reſpectively;

B, B′, B″, &c. for the 2d terms of R^{m-2}, R^{m-3}, R^{m-4}, &c.

C, C′, C″, &c. for the 3d terms of R^{m-3}, R^{m-4}, R^{m-5}, &c.

D, D′, D″, &c. for the 4th terms of R^{m-4}, R^{m-5}, R^{m-6}, &c. &c.

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Then it is evident, after expanding and multiplying, that

$$\left.\begin{array}{l} p\times(A+B+C+D+\&c.) \\ -2q\times(A'+B'+C'+D'+\&c.) \\ +3r\times(A''+B''+C'+D''+\&c.) \\ -4s\times(A'''+B'''+C'''+D''+\&c.) \\ \qquad \&c.\ \&c. \end{array}\right\} = a^m+b^m+c^m+d^m+\&c.$$

For the homologous terms muſt be equal, and the ſame power of x, viz. $\frac{1}{x^m}$ enters into every term, which therefore may be neglected.

We muſt next determine the values of A, A', A'', &c. B, B', B'', &c. C, C', C'', &c. &c. neglecting x, or calling it unity; then theſe values being ſubſtituted in the laſt equation, will give the ſum of the mth power of the roots.

By Emerſon's Algebra, page 151,

$$\left.\begin{array}{r} R^{m-1} = p^{m-1} - (m-1)\,qp^{m-2} + (m-1)\left(\frac{m-2}{2}\right) q^2p^{m-3} \\ + (m-1)\,rp^{m-2} \end{array}\right\}$$

$$\left.\begin{array}{r} -(m-1)\left(\frac{m-2}{2}\right)\left(\frac{m-3}{3}\right) q^3p^{m-4} \\ -(m-1)\left(\frac{m-2}{2}\right) 2qrp^{m-3} \\ -(m-1)\,sp^{m-2} \end{array}\right\}$$

$$\left.\begin{array}{r} +(m-1)\left(\frac{m-2}{2}\right)\left(\frac{m-3}{3}\right)\left(\frac{m-4}{4}\right) q^4p^{m-5} \\ +(m-1)\left(\frac{m-2}{2}\right)\left(\frac{m-3}{3}\right) 3q^2rp^{m-4} \\ +(m-1)\left(\frac{m-2}{2}\right)(2qs+r^2)p^{m-3} \\ +(m-1)\,tp^{m-2} \\ \&c. \end{array}\right\}$$

Hence by ſubſtituting $m-1$, $m-2$, $m-3$, &c. for $m-1$, the ſeveral values of A, A', A'', &c. B, B', B'', &c. C, C', C'', &c. will be found as follow.

$$A = p^{m-1},\quad B = -(m-2)\,qp^{m-3}$$
$$A' = p^{m-2},\quad B' = -(m-3)\,qp^{m-4},$$
$$A'' = p^{m-3},\quad B' = -(m-4)\,qp^{m-5},$$
$$\&c. \qquad \&c. \qquad \&c. \qquad \&c.$$

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$$C = \begin{cases} (m-3)\ rp^{m-4} \\ +(m-3)\left(\frac{m-4}{4}\right) q^2 p^{m-5} \end{cases}$$

$$C' = \begin{cases} (m-4)\ rp^{m-5} \\ +(m-4)\left(\frac{m-5}{2}\right) q^2 p^{m-6} \end{cases}$$

$$C'' = \begin{cases} (m-5)\ rp^{m-6} \\ +(m-5)\left(\frac{m-6}{2}\right) q^2 p^{m-7} \end{cases}$$

&c. &c. &c.

$$D = \begin{cases} -(m-4)\ sp^{m-5} \\ -(m-4)\left(\frac{m-5}{2}\right)\ 2qrp^{m-6} \\ -(m-4)\left(\frac{m-5}{2}\right)\left(\frac{m-6}{3}\right) qrp^{m-7} \end{cases}$$

$$D' = \begin{cases} -(m-5)\ sp^{m-6} \\ -(m-5)\left(\frac{m-6}{2}\right)\ 2qrp^{m-6} \\ -(m-5)\left(\frac{m-6}{2}\right)\left(\frac{m-7}{3}\right) qrp^{m-8} \end{cases}$$

$$D'' = \begin{cases} -(m-6)\ sp^{m-7} \\ -(m-6)\left(\frac{m-7}{2}\right)\ 2qrp^{m-8} \\ -(m-6)\left(\frac{m-7}{2}\right)\left(\frac{m-8}{3}\right) qrp^{m-9} \end{cases}$$

&c. &c.

$$E = \begin{cases} (m-5)\ tp^{m-6} \\ +(m-5)\left(\frac{m-6}{2}\right)\ (2qs+r^2)p^{m-7} \\ +(m-5)\left(\frac{m-6}{2}\right)\left(\frac{m-7}{3}\right)\ 3q^2rp^{m-8} \\ +(m-5)\left(\frac{m-6}{2}\right)\left(\frac{m-7}{3}\right)\left(\frac{m-8}{4}\right) q^3p^{m-8} \end{cases}$$

E' =

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$$E = \begin{cases} +(m-6)\,tp^{m-7} \\ +(m-6)\left(\frac{m-7}{2}\right)(2qs+r^2)\,p^{m-8} \\ +(m-6)\left(\frac{m-7}{2}\right)\left(\frac{m-8}{3}\right)3q^2rp^{m-9} \\ +(m-6)\left(\frac{m-7}{2}\right)\left(\frac{m-8}{3}\right)\left(\frac{m-9}{4}\right)q^4p^{m-10} \\ \&c. \end{cases}$$

&c.

Wherefore by substituting in the equation,

$$a^m+b^m+c^m+d^m = \begin{cases} p\times(A+B+C+D+\&c. \\ -2q\times(A'+B'+C'+D''+\&c. \\ +3r\times(A''+B''+C''+D''+\&c. \\ -4s\times(A'''+B'''+C'''+D'''+\&c\&c. \end{cases}$$

the values of A, B, C, &c. &c. found above, we shall have as follows

$$p\times(A+B+C+\&c.) = \begin{cases} p^m - (m-2)\,qp^{m-2} \\ \quad +(m-3)\,rp^{m-3} \\ \begin{cases} +(m-3)\left(\frac{m-4}{2}\right)q^2p^{m-4} \\ \quad -(m-4)\,sp^{m-4} \end{cases} \\ \begin{cases} \quad +(m-5)\,tp^{m-5} \\ -(m-4)\left(\frac{m-5}{2}\right)2qrp^{m-5} \end{cases} \\ \begin{cases} \quad -(m-6)\,up^{m-6} \\ +(m-5)\left(\frac{m-6}{2}\right)(2qs+r^2)p^{m-6} \\ -(m-4)\left(\frac{m-5}{2}\right)\left(\frac{m-6}{3}\right)q^3p^{m-6} \end{cases} \end{cases}$$

$$-2q\times(A'+B'+C'+\&c.) = \begin{cases} -2qp^{m-2} + (m-3)\,2q^2p^{m-4} \\ \quad -(m-4)\,2qrp^{m-5} \\ \begin{cases} -(m-4)\left(\frac{m-5}{2}\right)2q^3p^{m-6} \\ \quad +(m-5)\,2qsp^{m-6} \end{cases} \end{cases}$$

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$$+3^r \times (A'' + B^v + C^x + \&c.) = \left\{ \begin{matrix} 3^r p^{m-3} - (m-4) 3^q r p^{m-5} \\ + (m-5) 3^r s^2 p^{m-6} \end{matrix} \right\} \&c.$$

$$-4^s \times (A''' + B^{iv} + C^{vi} + \&c.) = -4^s p^{m-4} + (m-5) 4^q s p^{m-6} \&c.$$

$$+5^t \times (A^{iv} + B^{vi} + C^{iv} + \&c.) = +5^t p^{m-5} \&c.$$

$$-6^u \times (A^v + B^v + C^v + \&c.) = -6^u p^{m-6} \&c.$$

Hence by addition,

$$a^m + b^m + c^m + \&c. = \left\{ \begin{matrix} p^m - mqp^{m-2} + mrp^{m-3} \\ - ms \\ + m\left(\frac{m-3}{2}\right) q \end{matrix} \right\} p^{m-4} \left\{ \begin{matrix} + mt \\ - m(m-4) qr \end{matrix} \right\} p^{m-5} \left\{ \begin{matrix} - mu \\ + m(m-5) qs \\ - m\left(\frac{m-4}{2}\right)\left(\frac{m-5}{3}\right) qs \\ + m\left(\frac{m-5}{2}\right) r^2 \end{matrix} \right\} p^{m-6} \&c.$$

It is evident that the theorem may be carried to any number of terms by the method given above.

ARTICLE VIII.

To the Editor of the Repository.

SIR,

IF the following paper (for the contents of which I am indebted chiefly to Montucla's Hiſtoire des Mathematiques; and Bailly's Hiſtoire de L'Aſtronomie Moderne;) be thought worthy of a place in your Miſcellany, the inſertion of it will oblige,

Sir, your's, &c.

R. March 1ſt, 1802. L. R.

An attempt towards a Reſtitution of the loſt Treatiſe of Eratoſthenes, "de Locis ad Medietates".

ERATOSTHENES (borne at Cyrene 271 years before CHRIST;) was one of thoſe great and extraordinary men, whoſe minds embrace every

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every ſpecies of human knowledge :—He was called the ſurveyor of the univerſe, the Coſmographer ;—the ſecond Plato. His reputation for learning was ſo great, that he was invited from Athens to Alexandria, by Ptolemy Euergetes, and made by him keeper of the Royal Library at Alexandria, in which employment he died, in the 80th year of his age. It was at his request, that the ſame Ptolemy erected thoſe armillas, or circles of braſs, in the portico at Alexandria, with which Hipparchus, and Ptolemy, afterwards, ſo ſucceſsfully obſerved the heavenly bodies. He was the firſt who attempted to meaſure the Earth, and the method which he invented for this purpoſe, would, alone, have rendered his name immortal. He alſo invented the Aſtrolabe, with which he undertook to meaſure the obliquity of the ecliptic; and it is ſingular that the diſtance of the Sun from the Earth, which he eſtimated at 804,000,000 ſtadia, is exactly conformable to the diſtance aſſigned by Caſſini, and De la Caille. The works, that remain, of this excellent poet, grammarian, mathematician, and aſtronomer, were printed in an 8vo. volume, at Oxford, in the year 1672, and again at Amſterdam, in 1703.

In the mathematics, his attention was principally directed to geometry and aſtronomy : He indeed deſerved to be ranked with the celebrated geometricians of antiquity, *Ariſtæus*, *Euclid*, and *Apollonius*, who had written on geometrical analyſis :—*Pappus*, in the preface to his 7th book, mentions a treatiſe of his (now loſt,) in two books, entitled, "de Loci de Medietates", the object of which was to promote that analyſis; it is greatly to be lamented, that Pappus has left us no abſtract of this work; he has given a ſketch of the contents of ſeveral geometrical treatiſes, that are now loſt, but he has done little more than announce the title of this.

Moſt of the loſt treatiſes, of the ancient geometricians, have been reſtored by the moderns, but the reſtitution of *this* has never yet been attempted, by any perſon, that I know of: I ſhall however in what follows give a brief ſketch of what appears to me to have formed the two principal propoſitions in this work, leaving the taſk of demonſtrating them, to others.

It muſt be obſerved, in the firſt place, that, according to Pappus, theſe "*Loci ad Medietates*," were *Conic Sections*; and farther, that, with the ancients, *Mediatate*, (or *Mediety*,) was a *general term*, invented to expreſs three Lines, having either an arithmetical, a geometrical, or an *harmonical* relation; but the words *proportion*, (or ratio,) and analogy were reſtricted to *geometrical relation* only.

This being premiſed, I think it is extremely probable, that the two following theorems have been the ſubject of the loſt work of Eratoſthenes;—If I am miſtaken, I hope my readers will at leaſt give me credit for having noticed two properties of the conic ſections

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ſections which are not mentioned by any author I have read on that ſubject.

THEOREM I. *Fig.* 542, *Plate* xxix.

Let there be two right lines AB, CD, any how drawn, let the right line GH, be taken as an axis, and draw the line O F, making any angle whatever with G H; then, if the point P, (or p, or π, &c.) be ſo ſituated, that O P, (or $O p$, or $O \pi$, &c.) ſhall be with reſpect to O F, O E, in any continued arithmetical, geometrical, or harmonical mediety, whatever, *i. e.*

if OP : OE :: O E : O F, or if OE : O p :: O p : OF, or if OE : OF :: OF : O π; be in any arithmetical, geometrical, or harmonical proportion whatever; I ſay, that the points P (or p, or π, ſimilarly ſituated,) ſhall be in a Conic Section.

THEOREM II. *Fig.* 543, *Plate* xxix.

Let three right lines, AB, CD, GH, be drawn; let any axis LGA be taken at pleaſure; and draw the right line OFK, making any angle, whatever with CA; then, if the point P be ſo ſituated, that OP : OE :: OF : OI, in any continued arithmetical, geometrical, or harmonical mediety, whatever; I ſay, that all the points P, &c. ſo ſituated, ſhall be in a Conic Section. And it will be the ſame with the points p, π, $\tilde{\omega}$, &c. i. e.

if OE : OP :: OF : OI; or if OE : OF :: Oω : OI; or if OE : OF :: OI : O$\tilde{\omega}$, be in any continued arithmetical, geometrical, or harmonical proportion, whatever; then will the points p, π, and $\tilde{\omega}$ be in a Conic Section.

These Theorems may be eaſily demonſtrated by means of the modern analyſis; for it is evident that the dimenſions of the co-ordinates can never exceed the ſecond degree. It is alſo evident, that, if they were treated after the manner of the ancients, and if the different caſes which they admit of, were all ſeparately examined and investigated, they would afford ample matter for two books, one of which might contain the *firſt*, its caſes, corollaries, &c. and the other the *ſecond*.

For a further account of Eratoſthenes, and his writings, ſee *Vol. 1ſt* of Montucla's Hiſt. des Math. before mentioned, page 239—280, new edit. Paris 1799. And Bailly's Hiſt. de l'Aſtron. Moderne, B. I. Vol. I. And Phil. Tranſ. 1772.

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ARTICLE IX.

To the Editor of the Mathematical Repository.

SIR,

IF the following obſervations on the Fraction whoſe numerator and denominator vaniſh, ſhould be thought worthy of a place in your valuable Repoſitory, by inſerting the ſame you will greatly oblige,

Sir,

Your moſt obedient humble Servant,

Hull Citadel, Sept. 8, 1802. W. C.

Obſervations from a Periodical Publication.

" In this difficulty among mathematicians, I ſhall obſerve only, that when a fraction appears by aſſigning a peculiar value to the variable quantity under the form $\frac{0}{0}$, the fraction in that inſtance ceaſes to exiſt, and the ſuppoſition that it can have a determinate value, aroſe merely from inattention to the fraction in this ſuppoſed determinate ſtate.

" For example, $\frac{ax - x^2}{ax^2 - x^3}$ appear in the form $\frac{0}{0}$, when x is equal to a; conſequently, to uſe the common Rule, the fraction has in this caſe a determinate value, which is * $\frac{a\dot{x} - 2x\dot{x}}{2ax\dot{x} - 3x^2\dot{x}}$, or $\frac{a - 2x}{2ax - 3x^2}$, which, when x is equal to a, becomes $\frac{1}{a}$; that is, the fraction has the determinate value of $\frac{1}{a}$, when x is equal to a. Common ſenſe recoils at this determination, when it puts the algebraical terms into numbers, and making a equal to ten feet, and ſuppoſing x equal to it, the fraction appears to be $\frac{100 - 100}{1000 - 1000}$, or nothing at all; and on examining the fraction we ſee that it really is equal to $\frac{x \times (a - x)}{x^2 \times (a - x)}$, or $\frac{x}{x^2}$, or $\frac{1}{a}$, ſince $a - x$, divided by $a - x$, gives unity. But this ſuppoſed quo-

* The common rule for finding the value of a fraction whose numerator and denominator vanish, being to divide the fluxion of the numerator by the fluxion of the denominator.

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tient cannot be true, when $a - x$ ceases to be a number, and in that case the fraction is $\frac{1}{a} \times \frac{0}{0}$, or nothing at all, as it evidently ought to be. The fact shews itself at once to the person who would draw the curve corresponding to the equation $y = \frac{ax - x^2}{ax^2 - x^3}$, in which it is evident that all the ordinates will be greater than $\frac{1}{a}$; but as when x ceases to exist, no corresponding ordinate can be drawn to the curve, so no ordinate can be drawn when x becomes equal to a; and annihilates the relationship between y and x. The length of y extends without limit as the abscissa is decreased, it diminishes as x encreases, and its limit of decrease is $\frac{1}{a}$." Thus far the quotation.

Observations on the foregoing, by W. C.

In order to examine how far our writer's reasoning in the above paragraph may be consistent with mathematical demonstration, let us begin where he says, that the quotient of $a - x$ divided by $a - x$ cannot be equal to unity when $a - x$ ceases to be a number; now it is well known by every school boy that whilst $a - x$ remains a number, though infinitely small, the quotient of $a - x$ divided by $a - x$ must be equal to unity; and when $a - x$ ceases to be a number it is manifest that $\frac{a-x}{a-x} = \frac{0}{0} = 0^1 \div 0^1 = 0^{1-1} = 0^0$, this last expression has been pretty well handled at pages 398 and 399 vol 1. of the Math. Repository and there proved equal to 1; but at the same time the writer, on page 399 ibid. seems to hint that $\frac{0}{0}$ and 0^0 are not the same or equal, however my humble opinion is, that, from above it is very evident the two expressions are equal when 0 does actually represent no number at all, and consequently our question now is to find the value of 0^0 or $\frac{0}{0}$?

First for 0^0

Since the log. of $1 - z$ is $-z - \frac{1}{2}z^2 - \frac{1}{3}z^3 - \frac{1}{4}z^4 -$ &c. *ad infinitum* we shall have the log of $\overline{1-z}^{1-z}$ $= 1 - z$ into $-z - \frac{1}{2}z^2 - \frac{1}{3}z^3 - \frac{1}{4}z^4 - \frac{1}{5}z^5$ $= z^2 + \frac{1}{2}z^3 + \frac{1}{3}z^4 + \frac{1}{4}z^5 +$ &c. $- z - \frac{1}{2}z^2 - \frac{1}{3}z^3 - \frac{1}{4}z^4$ &c, which when z is equal to 1 will be

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$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$ &c. $- 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4}$ &c. $= 0$, whence the log. of $0^0 = 0$, and therefore 0^0 is equal to 1.

Secondly for $\frac{0}{0}$.—Becauſe the log. of $1 - 1 = 0$ is $-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4}$ —&c, the log. of $\frac{0}{0}$ will be $(-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} -$ &c.) minus $(-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} -$ &c$) = 0$, which is the log. of 1. Hence we ſee that 0^0 and $\frac{0}{0}$ are each equal to unity; and conſequently 0^0 is equal to $\frac{0}{0}$. Much more might be ſaid on this head but I ſhall leave it to abler hands.

Now to draw the curve correſponding to the equation $y = \frac{ax - x^2}{ax^2 - x^3}$; from what has been already done, it will manifeſtly appear that $y \left(= \frac{ax - x^2}{ax^2 - x^3}\right)$ will always be $= \frac{1}{x}$, for when $x = 0$, then $\frac{ax - x^2}{ax^2 - x^3} = \frac{1}{x} \times \frac{x}{x} \times \frac{a - x}{a - x}$ will be $= \frac{1}{0} \times \frac{0}{0} \times \frac{a - 0}{a - 0} = \frac{1}{0}$, becauſe $\frac{0}{0} = 1$; and when $x = a$, then $\frac{ax - x^2}{ax^2 - x^3} = \frac{1}{a} \times \frac{a}{a} \times \frac{0}{0} = \frac{1}{a}$, and for any other value of x than theſe two, it is evident that $y = \frac{1}{x}$, whence, making the ordinate (y) every where $= \frac{1}{x}$, a line being drawn through the extremity thereof will repreſent the curve required.—

How our writer ſhould imagine that the limit of y is $\frac{1}{a}$, I muſt leave to the judgment of the worthy and able correſpondents of the Mathematical Repoſitory, and ſincerely wish ſucceſs may attend the labours of every true lover of the Mathematics.

ARTICLE X.

Anſwer to Queſtion 238 propoſed in No. X, Tranſlated from Golovin's Ruſſian Edition of Euler's Theory on Ship-Building. Communicated by Mr. George Sanderſon.

QUESTION.

IF there be three arches or angles δ, n, ϕ, where δ is given,

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the expreſſion $\frac{\text{ſin. } (\delta - \eta - \phi)}{\text{ſin. } \varphi} \times \sqrt{\cos. \eta}$ will be the greateſt poſſible when the tan. $(\delta - \eta) = \frac{1}{2}$ tan. $\eta \times \frac{2 - \text{tan. } \eta \times \text{tan. } \varphi}{\frac{1}{2} - \text{tan. } \eta \times \text{tan, } \varphi}$; the relation between the angles η and φ being expreſſed by the equation, cot. $\eta = (a^3 \div 2b^3) \times$ tan. φ^2.

SOLUTION.

When any expreſſion is a maximum, its logarithm is a maximum alſo. The logarithm of

$\frac{\text{ſin. } (\delta - \eta - \varphi)}{\text{ſin. } \varphi} \times \sqrt{\cos. \eta} = \text{l. ſin. } (\delta - \eta - \phi) + \frac{1}{4}$ l. cos. η — l. ſin φ.

The fluxion of this being made $= 0$ gives

$\frac{\text{flux.}(\delta - \eta - \varphi) \times \cos.(\delta - \eta - \varphi)}{\text{ſin. } (\delta - \eta - \varphi)} - \frac{\dot{\eta} \times \text{ſin. } \eta}{2 \cos \eta} - \frac{\dot{\varphi} \times \cos. \varphi}{\text{ſin. } \varphi} = 0$:

But, δ being a conſtant quantity, by the queſtion, the fluxion of $(\delta - \eta - \varphi)$ will be $= -\dot{\eta} - \dot{\varphi}$; therefore

$\frac{-(\dot{\eta} + \dot{\varphi}) \times \cos. (\delta - \eta - \varphi)}{\text{ſin. } (\delta - \eta - \varphi)} - \frac{\dot{\eta} \times \text{ſin. } \eta}{2 \cos. \eta} - \frac{\dot{\varphi} \times \cos. \varphi}{\text{ſin. } \varphi} = 0.$

But cot. $\eta = \frac{a^3}{2b^3} \times$ tan. φ^2, therefore

l. cot. η = l. $\frac{a^3}{2b^3}$ + 2l. tan. φ; taking the fluxions,

$\frac{-\dot{\eta}}{\text{ſin. } \eta \times \cos. \eta} = \frac{2\dot{\varphi}}{\text{ſin. } \varphi \times \cos. \varphi}$; this being reduced gives,

$\dot{\eta} = -\frac{2\dot{\varphi} \times \text{ſin. } \eta \times \cos. \eta}{\text{ſin } \varphi \times \cos. \varphi}$: hence

$-(\dot{\eta} + \dot{\varphi}) = \frac{2\dot{\varphi} \times \text{ſin. } \eta \times \cos. \eta}{\text{ſin. } \varphi \times \cos. \varphi} - \dot{\varphi}$. Let theſe values of $\dot{\eta}$ and $-(\dot{\eta} + \dot{\varphi})$ be written in the expreſſion

$\frac{-(\dot{\eta} + \dot{\varphi}) \times \cos. (\delta - \eta - \varphi)}{\text{ſin. } (\delta - \eta - \varphi)} - \frac{\dot{\eta} \times \text{ſin. } \eta}{\cos. \eta} - \frac{\dot{\varphi} \times \cos. \varphi}{\text{ſin. } \varphi} = 0$, & we get

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$$\left.\begin{array}{l} \frac{2 \times \dot{\varphi} \times \text{fin. } \eta \times \cos. \eta \times \cos. (\delta - \eta - \varphi)}{\text{fin. } \varphi \times \cos. \varphi \times \text{fin. } (\delta - \eta - \varphi)} - \frac{\dot{\varphi} \times \cos. (\delta - \eta - \varphi)}{\text{fin. } (\delta - \eta - \varphi)} \\ + \frac{\dot{\varphi} \times \text{fin. } \eta^2}{\text{fin. } \varphi \times \cos. \varphi} - \frac{\dot{\varphi} \times \cos. \varphi}{\text{fin. } \varphi} \end{array}\right\} = 0, \text{ or}$$

$$\left.\begin{array}{l} \frac{2 \text{ fin. } \eta \times \cos. \eta \times \cos. (\delta - \eta - \varphi)}{\text{fin. } \varphi \times \cos. \varphi \times \text{fin. } (\delta - \eta - \varphi)} - \frac{\cos. (\delta - \eta - \varphi)}{\text{fin. } (\delta \cdot \eta - \varphi)} \\ + \frac{\text{fin. } \eta^2}{\text{fin. } \varphi \times \cos. \varphi} - \frac{\cos. \varphi}{\text{fin. } \varphi} \end{array}\right\} = 0.$$

Multiply this equation by $\frac{\text{fin. } (\delta - \eta - \varphi)}{\cos. (\delta - \eta - \varphi)} = \text{tan. } (\delta - \eta - \varphi)$. Then

$$\frac{2 \text{ fin. } \eta \times \cos. \eta}{\text{fin. } \varphi \times \cos. \varphi} - 1 + \frac{\text{fin. } \eta^2 \times \text{tan. } (\delta - \eta - \varphi)}{\text{fin. } \varphi \times \cos. \varphi} - \frac{\cos. \varphi \times \text{tan. } (\delta - \eta - \varphi)}{\text{fin. } \varphi} = 0$$

Make $P = \frac{2 \text{ fin. } \eta \times \cos. \eta}{\text{fin. } \varphi \times \cos. \varphi} - 1$ and $Q = \frac{\text{fin. } \eta^2}{\text{fin. } \varphi \times \cos. \varphi} - \frac{\cos. \varphi}{\text{fin. } \varphi}$.

Then $P + Q \times \text{tan. } (\delta - \eta - \varphi) = 0$.

But by trigonometry $\text{tan. } (\delta - \eta - \varphi) = \frac{\text{tan. } (\delta - \eta) - \text{tan. } \varphi}{1 + \text{tan. } (\delta - \eta) \times \text{tan. } \varphi}$.

Hence $P + Q \times \frac{\text{tan. } (\delta - \eta) - \text{tan. } \varphi}{1 + \text{tan. } (\delta - \eta) \times \text{tan. } \varphi} = 0$, or

$P + P \times \text{tan. } (\delta - \eta) \times \text{tan. } \varphi + Q \times \text{tan. } (\delta - \eta) - Q \times \text{tan } \varphi = 0$.

Hence $\text{tan. } (\delta - \eta) = \frac{Q \times \text{tan. } \varphi - P}{P \times \text{tan. } \varphi + Q}$. Let the values of P and Q be reſtored and we ſhall have

$$Q \times \text{tan. } \varphi - P = \frac{\text{fin. } \eta^2}{\cos. \varphi^2} - \frac{2 \text{ fin. } \eta \times \cos. \eta}{\text{fin. } \varphi \times \cos. \varphi}$$

$$= \frac{\text{fin. } \eta \times \cos. \eta}{\text{fin. } \varphi \times \cos. \varphi} \times (2 - \text{tan. } \eta \times \text{tan. } \varphi), \text{ and}$$

$$P \times \text{tan. } \varphi + Q = \frac{2 \text{fin. } \eta \times \cos. \eta}{\cos. \varphi^2} + \frac{\text{fin. } \eta^2}{\text{fin. } \varphi \times \cos. \varphi} - \frac{1}{\text{fin. } \varphi \times \cos. \varphi}.$$

But fin. $\eta^2 = 1 - \cos. \eta^2$; Therefore

$$P \times \text{tan. } \varphi + Q = \frac{2 \text{ fin. } \eta \times \cos. \eta}{\cos. \varphi^2} - \frac{\cos. \eta^2}{\text{fin. } \varphi \times \cos. \varphi}$$

$$= \frac{2 \cos. \eta^2}{\text{fin. } \varphi \times \cos. \varphi} \times (\tfrac{1}{2} - \text{tan. } \eta \times \text{tan. } \varphi).$$

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Hence tan. $(\delta - \eta) = \frac{Q \times \text{tan. } \varphi - P}{P \times \text{tan. } \varphi + Q}$

$= \frac{1}{2} \text{tan. } \eta \times \frac{2 - \text{tan.}^2 \eta \times \text{tan.}^2 \varphi}{\frac{1}{2} - \text{tan.}^2 \eta \times \text{tan.}^2 \varphi}$. *Q. E. D.*

ARTICLE XI.

Demonſtrations to Lawſon's Propoſitions propoſed in ARTICLES XXXVII and XLV, VOL. II.

PROP. XXXVII. Fig. 544, 545, Pl. 29.

Demonſtrated by Mr. John Lowry.

DRAW FH ⊥ to BC, and join FC. Then by Eu. 12 or 13, II,
$AF^2 = CF^2 + AC^2 + 2AC \cdot CH$, but
$CF^2 = AC \cdot CB$, by the hypotheſis; therefore,
$AF^2 = AC \cdot CB + AC^2 + 2AC \cdot CH$; but
$CB + AC = 2DC$; therefore
$AF^2 = 2AC \cdot DH = 2AC \cdot EF$. *Q. E. D.*

PROP. XXXVIII. Fig. 546, Pl. 29.

Demonſtrated by Mr. John Lowry.

Let ABCD be any regular figure circumſcribed about a circle whoſe centre is O. Let P be any point within the figure, and draw the perpendiculars PG, PH, PI, PK, &c. and join AP, BP, CP, DP, &c. Now it is evident that the ſum of all the triangles APB, BPC, CPD, APD, &c. or the rectangle under one of the ſides and half the ſum of the perpendiculars PG, PH, PI, PK, &c. is equal to the area of the regular figure ABCD, that is, equal to the multiple of the rectangle under one ſide and and the ſemi-diameter of the circle by the number of the ſides of the figure. Therefore the ſum of the perpendiculars will be equal to the multiple of the ſemi-diameter of the circle by the number of the ſides of the figure.

PROP. XXXIX. Fig. 547, Pl. 29.

Demonſtrated by Mr. Lowry.

Let the ſtraight lines AB, AC, AD, AE, &c. meet the circle

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circle in the points B, C, D, E, &c, and make the angles BAC, CAD, DAE, &c. equal. Then the arcs BC, CD, DE, &c. are equal. Join BD, and ED; then the angle BDE is equal to the angle BAF, that is, equal to the angle BAC. Therefore the arc BE is equal to the arc BC, and consequently the circle is divided into equal parts at the points B, C, D, E, &c.

PROP. XL. Fig. 548, 549, *Pl.* 29.

Demonstrated by Messrs. Cunliffe *and* Lowry.

In the triangle DEF draw DG making $\angle FDG = \angle D$, and meeting EF produced in G. Then $\angle FDG = \angle D$, that is by hypothesis $= \angle A$; also, by Euc. 32, I. $\angle DFG = \angle D + \angle E$, that is by hypothesis, $= \angle B$. Whence it is plain that the $\triangle$s ABC, DFG. are similar, therefore CA : CB :: GD : GF; but the $\angle GDE$ is bisected by DF; wherefore, by Euc. 3, VI. GD : GF :: DE : EF, and consequently CA : CB :: DE : EF *Q. E. D.*

The same, by Tyro Philomatheticus, *of Hull.*

On AB and DF demit the $\perp$s CR and ES; then by sim. $\triangle$s CA : CR :: ED : ES; but the $\angle CBA$ being $=$ to both the $\angle$s FDE and FED, the $\angle CBR$ must be $=$ to $\angle DFE$ (Simp. Geom. B. I. Prop. 1 and 9); therefore by sim. $\triangle$s CR : CB :: ES : EF, hence CA : CB :: ED : EF. *Q. E. D.*

PROP. XLI. Fig. 550, 551, 552. *Pl.* 29.

Demonstrated by Mr. James Cunliffe, *Fig.* 550.

Let ACB be a $\triangle$ having the vertical $\angle C$ bisected by the line CD meeting the base in D. Draw AE and BF $\perp$ to CD, and bisect CD in m, and in CD take CF $=$ DF.

It is evident that the $\triangle$s CAE and CBF, as well as the $\triangle$s AED and BFD are similar. Therefore

CE : CF :: AE : BF :: DE : DF, and alternately

CE : DE :: CF : DF; therefore by division

$CD = 2mD$: DE :: CF $-$ DF $= 2mF$: DF, or

mD : DE :: mF : DF, and by composition

mE : DE :: mD : DF; consequently

mD : mF :: mE : mD; Whence Euc. 17, VI.

$mF \cdot mE = (mD)^2$, or $2mE \cdot 2mF = 4\,(mD)^2 = CD^2$.

Now by the cor. to Prop. 24, B. II. of Emerson's Geom.

$BC^2 - BD^2 = 2mF \cdot CD$, and $AC^2 - AD^2 = 2mE \cdot CD$, whence

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$(BC^2 - BD^2)(AC^2 - AD^2) = 2mF \cdot 2mE \cdot CD^2 = CD^2 \cdot CD^2$, because $2mF \cdot 2mE = CD^2$, as deduced above; therefore
$BC^2 - BD^2 : CD^2 :: CD^2 : AC^2 - AD^2$. *Q. E. D.*

The same, by Mr. Lowry, *Fig.* 551.

Let CD, the line bisecting the vertical angle ACB of the triangle ABC, meet the base AB in D. Make AG, AE, each equal to AD, and BH, BF each equal to DB, and let DG, DH, DE, and DF be joined. Then $\angle AGD = ADG = \frac{1}{2}BAC$; therefore the $\angle CDG$ is half the supplement of the $\angle ABF$, that is, (because $\angle BDF = DFB$) $= \angle DFC$; therefore the $\triangle$s GDC, DCF are similar. In the same way it is shewn that the $\triangle$s HDC, DCE are similar. Therefore
GC : DC :: DC : CF, and EC : DC :: DC : CH, and compoundedly,
$GC \cdot EC : DC^2 :: DC^2 : CF \cdot CH$; but
$GC \cdot EC = AC^2 - AD^2$, and $CF \cdot CH = CB^2 - DB^2$; therefore
$AC^2 - AD^2 : DC^2 :: DC^2 :: CB^2 - DB^2$. *Q. E. D.*

The same by Tyro Philomatheticus, *Hull. Fig.* 552.

Make AF = to AD and $\perp$ to CF, and BE = to BD and $\perp$ to CE; then, CD being the line bisecting the vertical $\angle$, CF^2 will be = to $AC^2 - AD^2$, and $CE^2 = BC^2 - BD^2$.

Now by Simpson's Geom. B. IV. Prop. 16 and 18, the $\triangle$s AFC and BEC are similar, whence $\angle ECD = \angle FCD$;
again $\angle CED$ = a right-angle (CEB) + BED;
but $\angle BED = BDE = EDC + CDB$,
and the $\angle AFD = ADF$ = a right angle (AFG) — DFG;
therefore the $\angle$ FDG = a right-angle + DFG,
and the $\angle CDF = FDG + CDB$;
Now since the angles ECD and DCF are equal,
the $\angle CED + EDC = CDF + DFG$, that is,
a rt. $\angle$ + CDB + EDC + EDC = a rt. $\angle$ + CDB + DFG + DFG,
and taking away the $\angle$s that are common, we shall have
$2\angle EDC = 2DFG$; hence CED = CDF, and the $\triangle$s CED, and CDF are similar, therefore
CF : CD :: CD : CE, or $CF^2 : CD^2 :: CD^2 : CE^2$. *Q. E. D.*

PROP. XLII. Fig. 553, 554, 555. *Pl.* 29.

Demonstrated by Mr. James Cunliffe, *Fig.* 553.

Let FG be a right line intercepted by the tangents, cutting the circle in M and N, and bisected by the chord DE in *m*.

Draw GH parallel to DE meeting the tangent CF in H, and to S,

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to S, the centre of the circle, draw the radius DS; alſo draw the reſt of the lines as in the figure.

Becauſe mD and GH are parallel, and Fm = mG, it is plain that FD = DH: but SD is ⊥ to DF, by Euc. 18, III; therefore SF = SH = SG; conſequently the triangle FSG is iſoſceles. Now the line Sm, which biſects FG will be perpendicular thereto, as appears from Euc. 12, I; Sm will therefore biſect the chord MN by Euc. 3. III. *Q. E. D.*

The ſame by Mr. John Lowry, *Fig.* 554.

Let AB, AC, be two tangents drawn from the point A and touching the circle in B and C, and let EF be alſo drawn to meet the circle at I and H and the line joining BC at D. Then if FD be = to DE, ID will be = to DH.

Draw FG parallel to BC and let it meet AC produced in G, and join the centre O with the points E, B, F, G, C and D. Then ſince AB = AC, AE is = AG, and conſeq. OF = OG: but DE is = DF (by hypotheſis); therefore by parallels EC = CG, and OC is ⊥ to EG; therefore OE = OG = OF, and conſeq. OD is ⊥ to EF, and therefore DI is = DH.

Converſely. If DI be = DH, FD will be = to DE.

For DO is ⊥ to IH, therefore the points B, F, D, O, are in a circle, and ſo are alſo the points O, D, E, C; therefore the ∠ DEO = OCD = OBD = OFD, and OF = OE; therefore ED = DF. *Q. E. D.*

The ſame by Tyro Philomatheticus. *Fig.* 555.

The ∠ AFE being = BFD, and EAF = BFD + BDE,
we have by Prop. XL. FE : EA :: FD : DB,
but FE = FED, therefore EA = DB,
and by Simpſon's Geometry III. 32,
the rectangle GEH = EA², and HDG = DB²;
whence GE = HD, and GF = FH.

Converſely. By Prop. XL, FE : EA :: FD : DB,
FE² : EA² :: FD² : DB²,
but EA² = rectangle GEH, and DB² = HDG, therefore
EE² : GEH :: FD² : HDG, and by diviſion
FE² — GEH : GEH :: FD² — HDG : HDG; that is,
GF² : GEH :: HF² : HDG, but GF = HF; hence
GEH = HDG, or EA = DB, and conſeq. FE = FD. *Q. E. D.*

PROP.

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PROP. XLIII. Fig. 556. *Pl.* 29.

Demonſtrated by Meſſrs. Cunliffe, Lowry, *and* Tyro Philomatheticus.

Let ACD be a $\triangle$ having its ſides AC, DC, equal to each other.

Draw DE and C*m* $\perp$ to the ſides AC and AD reſpectively. It appears from Euc. 12. I. that AD is biſected in *m*. And becauſe D*m*C and DEC are right angles, the points D, *m*, E, C are in the circumference of the ſame circle, therefore by Euc. 36, III. $AC \cdot AE = AD \cdot Am = AD \cdot \frac{1}{2}AD = \frac{1}{2}AD^2$. *Q. E. D.*

PROP. XLIV. Fig. 557, *Pl.* 29.

Demonſtrated by Meſſrs. Cunliffe, Lowry, *and* Tyro Philomatheticus.

Firſt. $(AB+AD)\,BC+BC^2 = AB \cdot BC + AD \cdot BC + BC^2$
$= AB \cdot BC + (AD+BC) \cdot BC$
$= AB \cdot BC + (AB+CD) \cdot BC$
$= AB \cdot BC + AB \cdot BC + CD \cdot BC$
$= 2AB \cdot BC + CD \cdot BC$. *Q. E. D.*

Secondly. $(AB+AD) \cdot CD + CD^2 = (2AD+DB) \cdot CD + CD^2$
$= 2AD \cdot CD + DB \cdot CD + CD^2$
$= 2AD \cdot CD + (DB+CD) \cdot CD$
$= 2AD \cdot CD + BC \cdot CD$. *Q.E.D.*

PROP. XLV. Fig. 558, 559, *Pl.* 29.

Demonſtrated by Mr. John Lowry.

Let ABC be a $\triangle$, and let CD, CE be drawn to make the $\angle$s ACD, ECB equal. Through the points D, E, C, deſcribe a circle to meet AC, CB in F and G, and join FG. Then becauſe the $\angle$s FCD, ECB are equal, the arcs FD, GF are alſo equal, therefore FG is parallel to AB. Wherefore

AC : BC :: AF : BG, or

$AC^2 : BC^2 :: AC \cdot AF = AD \cdot AE : BC\,BG = BE \cdot BD$. *Q. E. D.*

PROP. XLVI. Fig. 560, 561, 562, 563, 564. *Pl.* 29.

Demonſtrated by Mr. James Cunliffe, *Fig.* 560, 561.

Firſt. When the point C is without the circle.

Draw

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Draw AD, upon which produced let fall the ⊥ CF and join BD. The ∠ ADB in a ſemi-circle is a right angle, by Euc. 31, III. wherefore the triangles AED and ADB are ſimilar; therefore AE : AD :: AD : AB, whence by Euc. 17, VI.

$AD^2 = AE \cdot AB$. And becauſe CFD and CED are right angles, the points C, E, D, F, lie in the circumference of a circle as appears from Euc. 22, III; therefore by Euc. 36, III.

$AC \cdot AE = AF \cdot AD$: but

$AC \cdot AE = (AB + BC) \cdot AE = AB \cdot AE + BC \cdot AE$
$= AD^2 + BC \cdot AE$, and

$AF \cdot AD = (AD + DF) \cdot AD = AD^2 + AD \cdot DF$;

therefore $AD^2 + BC \cdot AE = AD^2 + AD \cdot DF$, and

by taking away the common ſquare AD^2 from each, there will remain

$BC \cdot AE = AD \cdot DF$: but from what is juſt deduced,

$AC \cdot AE = AD^2 + AD \cdot DF$, whence by addition

$AC \cdot AE + BC \cdot AE = AD^2 + 2AD \cdot DF$, or

$(AC + BC) \cdot AE = AD^2 + 2AD \cdot DF$; ∴ by Euc. 12, II.

$AC^2 = DC^2 + AD^2 + 2AD \cdot DF = DC^2 + (AC^2 + BC^2) \cdot AE$.

Q. E. D.

Secondly. When the point C is within the circle.

The triangles AED and ADB are ſimilar, therefore

AE : AD :: AD : AB; whence

$AD^2 = AB \cdot AE = (AC + BC) \cdot AE = AC \cdot AE + BC \cdot AE$, and

$AD \cdot AF = AC \cdot AE$; therefore

$AD^2 - AD \cdot AF = (AD - AF) \cdot AD$
$= AD \cdot DF = BC \cdot AE$, or
$2AD \cdot DF = 2BC \cdot AE$: but

$AD^2 = AC \cdot AE + BC \cdot AE$, as deduced above. Whence

$AD^2 - 2AD \cdot DF = AC \cdot AE - BC \cdot AE$
$= (AC - BC) \cdot AE$, and theref. Eu. 13, II.

$AC^2 = DC^2 + AD^2 - 2AD \cdot DF = DC^2 + (AC - BC) \cdot AE$.

Q. E. D.

The ſame by Mr. John Lowry. *Fig.* 562, 563.

$AC^2 = CE^2 + AE^2 + 2CE \cdot EA$; but

$AE^2 + 2CE \cdot AE = (AE + 2CE) \cdot AE = (AC + CB + BE) \cdot AE$ *fig.* 562
or $= (AC - CB + BE) \cdot AE$, *fig* 563.

Alſo $CD^2 = CE^2 + ED^2$, and $ED^2 = BE \cdot EA$;

Therefore $CD^2 = CE^2 + BE \cdot EA$;

Conſequently $AC^2 = CD^2 + (AC \pm CB) \cdot AE$.

Q. E. D.

The

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The same by Tyro Philomatheticus. *Fig.* 564.

It is evident that
$AC^2 (=CE^2 + AC^2 - CE^2) = CE^2 + (AC - CE) \cdot (AC + CE)$;
now when C is in the diameter produced
$AC - CE = AE$, and $AC + CE = AC + CB + EB$; whence
$AC^2 = CE^2 + (AC + CB + EB) \cdot AE$
$= CE^2 + AE \cdot EB + AE \cdot (AC + CB)$
$= CD^2 + AE \cdot (AC \cdot CB)$;
and when C is in the diameter itself,
$AC + CE = AE$, and $AC - CE = AC - CB + EB$; therefore
$AC^2 = CE^2 + AE \cdot (AC - CB + EB)$
$= CE^2 + AE \cdot EB + AE \cdot (AC - CB)$
$= CD^2 + AE \cdot (AC - CB)$. *Q. E. D.*

ARTICLE XII.

Demonstration that the Area of a Parabolic Segment contained under the Abscissa, the ordinate and the curve is equal to two-thirds of the parallelogram contained under the abscissa and ordinate. By WILLIAM FREND, Esq. Fellow of Jesus College, Cambridge.

LET ABC (*Fig.* 578, *Plate* 29.) be a segment of a parabola under the abscissa AB, ordinate BC, and curve AC. Through C draw CD parallel to AB, and through A draw AD parallel to BC. Then the two curvilinear areas ABC, ACD, make up the whole parallelogram ABCD; and if p and q are certain fractional numbers

$$ABC = \frac{ABCD}{p} \text{ and } ACD = \frac{ABCD}{q}.$$

Let the lines AB, AD, be produced to E and I, and through G draw GI parallel to AE and meeting BC produced in H. And join C, G.

Then when the abscissa AB is increased by the magnitude BE and the ordinate BC by the magnitude FG, the area ABC is increased by the mixtilineal area BEGC, and in the same manner the area ACD is increased by the mixtilineal area DCGI. Let the area intercepted between the chord CG and the arc CG be called v; the line $AB = x$, the ordinate $BC = y$, the increment of $AB = m = BE$ and the increment of $BC = n = FG$.

Then

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Then the mixtilineal area BCGE $= ym + \frac{mn}{2} + v$,

and the mixtilineal area DCGI $= xn + \frac{mn}{2} - v$.

Let the parameter $= 2a$,

therefore $y^2 = 2ax$ and $x = \frac{y^2}{2a}$,

and $(y+n)^2 = 2a \times (x+m)$;
therefore $y^2 + 2yn + n^2 = 2ax + 2am$,
or $2yn + n^2 = 2am$

and $\frac{yn}{a} + \frac{n^2}{2a} = m$.

Therefore $ym = \frac{y^2n}{a} + \frac{yn^2}{2a}$

and $\frac{mn}{2} = \frac{yn^2}{2a} + \frac{n^3}{4a}$;

Therefore $ym + \frac{mn}{2} + v = \frac{y^2n}{a} + \frac{yn^2}{2a} + \frac{n^3}{4a} + v =$ BCGE

and $xn + \frac{mn}{2} - v = \frac{y^2n}{2a} + \frac{yn^2}{2a} + \frac{n^3}{4a} - v =$ DCGI.

Now $\frac{\text{AEGI}}{p} - \frac{\text{ABCD}}{p} = \frac{(x+m)(y+n)}{p} - \frac{xy}{p}$

$= \frac{(y+n)^2}{2a} \times \frac{y+n}{p} - \frac{y^2}{2a} \times \frac{y}{p}$

$= \frac{(y+n)^3}{2ap} - \frac{y^3}{2ap}$

$= \frac{3y^2n}{2ap} + \frac{3yn^2}{2ap} + \frac{n^3}{2ap} =$ A

Make A $=$ BCGE;

Therefore $\frac{y^2n}{a} + \frac{yn^2}{a} + \frac{n^3}{4a} + v = \frac{3y^2n}{2ap} + \frac{3yn^2}{2ap} + \frac{n^3}{2ap}$;

now if it is true that the parabolic area AEG is equal to $\frac{\text{AEGI}}{p}$, then since it will be true whatever is the magnitude of m, the homologous terms on each side of the equation will be equal to each other *.

* See Preface to my Principles of Algebra, or the true Theory of Equations established on Mathematical Demonstration. Part the Second, page xii.

Namely

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Namely $\frac{y^2n}{a} = \frac{3y^2n}{2ap}$, or $p = \frac{3}{2}$,

$\frac{yn^2}{a} = \frac{3yn^2}{2ap}$, or $p = \frac{3}{2}$,

$\frac{n^3}{4a} + v = \frac{n^3}{2ap} = \frac{n^3}{3a}$;

Therefore $v = \frac{n^3}{3a} - \frac{n^3}{4a} = \frac{n^3}{12a}$.

Alſo $\frac{AEGI}{q} - \frac{ABCD}{q} = \frac{3y^2n}{2aq} + \frac{3yn^2}{2aq} + \frac{n^3}{2aq} = B$.

Make $B = DCGI$.

Therefore $\frac{y^2n}{2a} + \frac{yn^2}{2a} + \frac{n^3}{4a} - v = \frac{3y^2n}{2aq} + \frac{3yn^2}{2aq} + \frac{n^3}{2aq}$.

Now if it is true that the area ACGI is equal to $\frac{AEGI}{q}$, then ſince it will be true whatever is the magnitude of n the homologous terms on each ſide of the equation will be equal to each other*.

Namely $\frac{y^2n}{2a} = \frac{3y^2n}{2aq}$, or $q = 3$

$\frac{yn^2}{2a} = \frac{3yn^2}{2aq}$, or $q = 3$

$\frac{n^3}{4a} - v = \frac{n^3}{2aq} = \frac{n^3}{6a}$;

Therefore $v = \frac{n^3}{4a} - \frac{n^3}{6a} = \frac{n^3}{12a}$ as before.

Thus the numbers p and q are determined; namely p to be $\frac{3}{2}$ and q to be 3, conſequently the area $ABC = \frac{2}{3}$ ABCD, and the area $ACD = \frac{1}{3}$ ABCD.

Cor. The area contained between the chord of a parabola and the arc of the parabola may always be found. And if the chord is drawn from the vertex of the parabola this area is equal to the ſquare of the ordinate multiplied into the number, found by dividing the ordinate by twelve times the ſemi-parameter.

*. See the preceding Note.

ARTICLE

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ARTICLE XIII.

Improved Solutions to ſome curious Mathematical Problems.

(Continued from Page 10.)

PROB. III. *By Mr.* JAMES CUNLIFFE.

TO find three poſitive integers, ſuch, that the difference of every two of them may be a ſquare number; and alſo that the difference between the ſum of every two of them and the third may be a ſquare number?

SOLUTION.

Denote the required integers by x, y, and z; and by the queſtion, put $x+y-z=a^2$, $x+z-y=b^2$, and $y+z-x=c^2$.

Half the ſum of the firſt and ſecond of theſe is $\left.\begin{matrix}\\\\\end{matrix}\right\} x=\frac{1}{2}(a^2+b^2)$;

Half the ſum of the firſt and third is $y=\frac{1}{2}(a^2+c^2)$;

And half the ſum of the ſecond and third is $z=\frac{1}{2}(b^2+c^2)$.

From whence $x-y=\frac{1}{2}(b^2-c^2)$, $a-z=\frac{1}{2}(a^2-c^2)$, and $y-z=\frac{1}{2}(a^2-b^2)$; which three expreſſions muſt all be ſquare numbers by the queſtion; in order to effect this, put

$x-y=\frac{1}{2}(b^2-c^2)=e^2$, and $x-z=\frac{1}{2}(a^2-c^2)=d^2$; from whence $y-z=\frac{1}{2}(a^2-b^2)=d^2-e^2=$ a ſquare by the queſtion.

From the laſt expreſſion $a^2-b^2=2(d^2-e^2)$, which may be divided into factors thus $(a+b)(a-b)=\frac{2s}{r}(d+e)\times\frac{r}{s}(d-e)$.

Put $a+b=\frac{2s}{r}(d+e)$, then will $a-b=\frac{r}{s}(d-e)$;

from which two equations we have

$$a=\frac{s}{r}(d+e)+\frac{r}{2s}(d-e)=\frac{d(2s^2+r^2)+e(2s^2-r^2)}{2rs}, \text{ and}$$

$$b=\frac{s}{r}(d+e)-\frac{r}{2s}(d-e)=\frac{d(2s^2-r^2)+e(2s^2+r^2)}{2rs}.$$

But d^2-e^2 muſt be a ſquare as obſerved above, and it is well known that this will be the caſe when $d=2rs(m^2+n^2)$, and $e=2rs\times 2mn$, theſe being ſubſtituted for d and e in the above expreſſions for a and b they become

$a=(m^2+n^2)(2s^2+r^2)+2mn(2s^2-r^2)$, and

$b=(m^2+n^2)(2s^2-r^2)+2mn(2s^2+r^2)$. Now, by means of theſe and the preceding part of the ſolution

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$$c^2=b^2-2c^2=\left\{(n^2+n^2)(2s^2-r^2)+2mn(2s^2+r^2)\right\}^2-32r^2s^2m^2n^2$$
$$=\left\{\begin{array}{l}(m^2+n^2)^2(2s^2-r^2)^2+4mn(m^2+n^2)(4s^4-r^4)\\+4m^2n^2(2s^2+r^2)^2-32r^2s^2m^2n^2\end{array}\right.$$
$$=\left\{\begin{array}{l}(m^2+n^2)^2(2s^2-r^2)^2+4mn(m^2+n^2)(4s^4-r^4)\\+4m^2n^2(2s^2-r^2)^2.\end{array}\right.$$

Dividing by $(2s^2-r^2)^2$ there will be had

$$\frac{c^2}{(2s^2-r^2)^2}=(m^2+n^2)^2+4mn(m^2+n^2)\left(\frac{2s^2+r^2}{2s^2-r^2}\right)+4m^2n^2$$
$$=m^4+4qm^3n+6m^2n^2+4qmn^3+n^4, \text{ by putting } q=\frac{2s^2+r^2}{2s^2-r^2}$$

Assume $m^2+2qmn-n^2=c\div(2s^2-r^2)$, then will

$$\frac{c^2}{(2s^2-r^2)^2}=(m^2+2qmn-n^2)^2$$
$$=m^4+4qm^3n+4q^2m^2n^2-2m^2n^2-4qmn^3+n^4$$
$$=m^4+4qm^3n+6m^2n^2+4qmn^3+n^4.$$

Whence $4qm^2n^2-2m^2n^2-4qmn^2=6m^2n^2+4qmn^2$,
or $m^2n^2(4q^2-8)=8qmn^2$; Dividing by $4mn^2$

$$m(q^2-2)=n(2q), \text{ or } \frac{m}{n}=\frac{2q}{q^2-2}=\frac{2(4s^4-r^4)}{12s^2r^2-4s^4-r^4}.$$

Whence it appears that m and n may be denoted by any two numbers in the ratio of $2(4s^4-r^4)$ and $12s^2r^2-4s^4-r^4$ respectively, and r and s taken at pleasure.

Now $\frac{c}{2s^2-r^2}=m^2+2qmn-n^2=m^2-n^2+2\left(\frac{2s^2+r^2}{2s^2-r^2}\right)mn$, as assumed above, whence $c=(m^2-n^2)(2s^2-r^2)+2mn(2s^2+r^2)$.

Example. If r and s be taken each $=1$, then m and n will be found to be in the ratio of 6 to 7, or we may put $m=6$, and $n=7$:

then $a=(m^2+n^2)(2s^2+r^2)+2mn(2s^2-r^2)=339$,
$b=(m^2+n^2)(2s^2-r^2)+2mn(2s^2+r^2)=337$, and
$c=(m^2-n^2)(2s^2-r^2)+2mn(2s^2+r^2)=239$.

Whence $x=\frac{1}{2}(a^2+b^2)=114245$, $y=\frac{1}{2}(a^2+c^2)=86021$, and $z=\frac{1}{2}(b^2+c^2)=85345$, which are three numbers that will answer the conditions of the question.

PROB. IV. *By Mr.* JAMES CUNLIFFE.

To find three square whole numbers, such, that the difference between the sum of every two of them, and the third may be a square number.

SOLUTION

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Solution.

Let the required ſquares be denoted by x^2, y^2, and z^2, and by the queſtion, put

$x^2+y^2-z^2=a^2$, $x^2+z^2-y^2=b^2$, and $y^2+z^2-x^2=c^2$.

The ſum of the three equations is $x^2+y^2+z^2=a^2+b^2+c^2$, and by taking half the difference between this and each of the former equations there will be had

$x^2=\frac{1}{2}(a^2+b^2)$, $y^2=\frac{1}{2}(a^2+c^2)$, and $z^2=\frac{1}{2}(b^2+c^2)$;

from the two firſt of theſe $a^2=2x^2-b^2=2y^2-c^2$.

Put $x=y+rv$, and $b=c+sv$, then

$$2x^2-b^2=2(y+rv)^2-(c+sv)^2$$
$$=2y^2-c^2+4ryv+2r^2v^2-2scv-s^2v^2=2y^2-c^2.$$

Whence $v=(2sc-4ry)\div(2r^2-s^2)$;

therefore $x=y+rv=\left\{2rsc-y(2r^2+s^2)\right\}\div(2r^2-s^2)$,

and $b=c+sv=\left\{c(2r^2+s^2)-4rsy\right\}\div(2r^2-s^2)$.

Now from what is done $2y^2-c^2=a^2$, and therefore that y, c, and a may be rational, put $(2r^2-s^2)(m^2+n^2)=y$,

$(2r^2-s^2)(n^2-m^2+2nm)=c$, and $(2r^2-s^2)(m^2-n^2+2nm)=a$.

The reaſon of theſe laſt aſſumptions will perhaps appear plainer by obſerving that a, y, and c are the roots of three ſquares in arithmetical progreſſion. By means of which, and what is before deduced

$x=2rs(n^2-m^2+2mn)-(2r^2+s^2)(m^2+n^2)$, and

$b=(2r^2+s^2)(n^2-m^2+2mn)-4rs(m^2+n^2)$: therefore

$$z^2=\tfrac{1}{2}(b^2+c^2)=\left\{\begin{array}{l}(4r^4+s^4)(n^2-m^2+2mn)^2+4rs(2r^2+s^2)(m^2-n^2)\\-8rsmn(2r^2+s^2)(m^2+n^2)+8r^2s^2(m^2+n^2)^2\end{array}\right.$$
$$=\left\{\begin{array}{l}m^4(2r^2+2rs+s^2)^2-4m^3n(4r^4+4r^3s+2rs^3+s^4)\\+2m^2n^2(4r^4+8r^2s^2+s^4)+4mn^3(4r^4-4r^3s-2rs^3+s^4)\\+n^4(2r^2-2rs+s^2)^2.\end{array}\right.$$

Aſſume $\left\{\begin{array}{l}m^2(2r^2+2rs+s^2)\\-2mn\left(\dfrac{4r^4+4r^3s+2rs^3+s^4}{2r^2+2rs+s^2}\right)\\-n^2(2r^2+2rs+s^2)\end{array}\right\}=z$, then will

$$z^2=\left\{\begin{array}{l}m^4(2r^2+2rs+s^2)^2-4m^3n(4r^4+4r^3s+2rs^3+s^4)\\+2m^2n^2(4r^4+8r^2s^2+s^4)\\+4mn^3(4r^4-4r^3s-2rs^3+s^4)+n^4(2r^2-2rs+s^2)^2.\end{array}\right.$$

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$$= \begin{cases} m^4 (2r^2+2rs+s^2)^2 - 4m^3n (4r^3+4r^2s+2rs^2+s^3) \\ + 4m^2n^2 \left(\frac{4r^3+4r^2s+2rs^2+s^3}{2r^2+2rs+s^2}\right)^2 \\ - 2m^2n^2 (4r^4+s^4) \\ + 4mn^3 (4r^3+4r^2s+2rs^2+s^3) \left(\frac{2r^2-2rs+s^2}{2r^2+2rs+s^2}\right) \\ + n^4 (2r^2-2rs+s^2)^2. \end{cases}$$

This being reduced gives

$$m = n \times \frac{(4r^4+s^4)(4r^4+4r^3s-2rs^3+s^4)-(2r^2+2rs+s^2)(4r^4-4r^3s-2rs^3+s^4)}{(2r^2+s^2)^2(2r^2+2rs+s^2)^2-(4r^4+4r^3s+2rs^3+s^4)^2},$$

which may be farther reduced to

$$m = n \times \frac{4r^3s+4r^2s^2+2rs^3}{4r^4+4r^3s+2r^2s^2+2rs^3+s^4},$$

$$\text{or } \frac{m}{n} = \frac{4r^3s+4r^2s^2+2rs^3}{4r^4+4r^3s+2r^2s^2+2rs^3+s^4}.$$

Hence it appears that m and n may be expounded by any two numbers in the ratio of $4r^3s+4r^2s^2+2rs^3$ and $4r^4+4r^3s+2r^2s^2+2rs^3+s^4$ reſpectively, and r and s taken at pleaſure.

Example 1. Suppoſe r and s each $= 1$, then $\frac{m}{n} = \frac{10}{13}$, whence we may take $m = 10$ and $n = 13$, by means of which,

$x = 2rs(n^2-m^2+2mn) - (r^2+s^2)(m^2+n^2) = 149$

$y = (2r^2-s^2)(m^2+n^2) = 269$, and

$$z = m^2(2r^2+2rs+s^2) - 2mn\left(\frac{4r^4+4r^3s+2rs^3+s^4}{2r^2+2rs+s^2}\right) - n^2(2r^2-2rs+s^2) = -241.$$

or 149, 269, and 241 are the roots of three ſquares that will anſwer.

Example 2. If $r = 1$ and $s = -1$, then $\frac{m}{n} = \frac{-2}{1}$, or $m = -2$ and $n = 1$; whence $x = -1$, $y = 5$, and $z = -5$, or the roots of the ſquares may be 1, 5 and 5, but as two of them are equal they deſerve little notice.

I ſhall now ſhew how the expreſſions for x and b might have been obtained by the method of factors.

It was found that $a^2 = \frac{1}{2}(2x^2-b^2) = \frac{1}{2}(2y^2-c^2)$, whence $x^2-y^2 = \frac{1}{2}(b^2-c^2)$; this expreſſion may be divided into factors thus $(x+y)(x-y) = \frac{r}{2s}(b+c) \times \frac{s}{r}(b-c)$;

put $x+y = \frac{r}{2s}(b+c)$, then will $x-y = \frac{s}{r}(b-c)$; whence

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$$x = \frac{r}{4s}(b+c) + \frac{s}{2r}(b-c) = \frac{r^2(b+c) + 2s^2(b-c)}{4rs}$$
$$= \frac{b(2s^2+r^2) - c(2s^2-r^2)}{4rs}, \text{ and}$$

$$y = \frac{r}{4s}(b+c) - \frac{s}{2r}(b-c) = \frac{r^2(b+c) - 2s^2(b-c)}{4rs}$$
$$= \frac{c(2s^2+r^2) - b(2s^2-r^2)}{4rs}, \text{ and}$$

from the laſt of theſe $b = \frac{c(2s^2+r^2) - 4rsy}{2s^2-r^2}$.

But $y^2 = \frac{1}{2}(a^2+c^2)$ or $2y^2 = a^2+c^2$, that is a^2, y^2, and c^2, are three ſquares in arithmetical progreſſion; therefore we may put $(2s^2-r^2)(m^2-n^2+2mn) = a$, $(2s^2-r^2)(m^2+n^2) = y$, and $(2s^2-r^2)(n^2-m^2+2mn) = c$; by means of which

$$b = \frac{c(2s^2+r^2) - 4rsy}{2s^2-r^2} = (2s^2+r^2)(n^2-m^2+2mn) - 4rs(m^2+n^2), \text{and}$$

$$x = \frac{b(2s^2+r^2) - c(2s^2-r^2)}{4rs} = 2rs(n^2-m^2+2mn) - (2s^2+r^2)(m^2+n^2),$$

which are the ſame expreſſions as were found the other way.

An anſwer to the preceding queſtion may be ſeen in the Gentleman's Diary for 1802, which differs from that here given only in this, viz. in the Diary, the values of x, y, and z, are deduced from thoſe of a, b, and c, previously obtained, which is not neceſſary in the calculation.

PROB. V. *By Mr.* JAMES CUNLIFFE.

To find the value of x ſo that x^2+ax, x^2+bx, and x^2+cx may be all rational ſquares.

SOLUTION.

Aſſume $n-x$ for the root of the firſt; that is, make

$x^2+ax = (n-x)^2 = n^2-2nx+x^2$, from whence $= x\ \frac{n^2}{2n+a}$;

this being written for x in the ſecond condition gives,

$$x^2+bx = \frac{n^4}{(2n+a)^2} + \frac{bn^2}{2n+a} = \frac{n^2}{(2n+a)^2} \times \left\{ n^2+b(2n+a) \right\} =$$

a ſquare; therefore $n^2+b(2n+a) =$ a ſquare

 Aſſume

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Assume $s - n$ for the root of this square, and then

$n^2 + b(2n+a) = (s-n)^2 = s^2 - 2sn + n^2$, whence $n = \frac{s^2 - ab}{2(b+s)}$,

and $x = \frac{n^2}{2n+a} = \frac{(s^2-ab)^2(b+s)}{4(b+s)^2(s^2+sa)} = \frac{(s^2-ab)^2}{4s(a+s)(b+s)}$; and this being substituted for x in the third and last condition gives

$$x^2 + cx = \frac{(s^2-ab)^4}{16s^2(a+s)^2(b+s)^2} + \frac{c(s^2-ab)^2}{4s(a+s)(b+s)}$$

$$= \frac{(s^2-ab)^2}{16s^2(a+s)^2(b+s)^2} \times \left\{(s^2-ab)^2 + 4cs(a+s)(b+s)\right\} =$$

a square;

therefore $(s^2-ab)^2 + 4cs(a+s)(b+s)$ $\Big\} = \Big\{ a^2b^2 + 4sabc + 2s^2\left\{2c(b+a) - ab\right\} + 4cs^3 + s^4 = \text{a square.}$

Assume $ab + 2cs - s^2$ for the root of the last square, then will

$$\left.\begin{array}{l} a^2b^2 + 4sabc + 2s^2\left\{2c(b+a) - ab\right\} \\ + 4cs^3 + s^4 \end{array}\right\} = (ab + 2cs - s^2)^2$$

$$= \left\{\begin{array}{l} a^2b^2 + 4sabc + 4c^2s^2 \\ - 2abs^2 - 4cs^3 + s^4; \end{array}\right.$$

Whence $2s^2 \times 2c(b+a) + 4cs^3 = 4c^2s^2 - 4cs^3$, and by transposition $8cs^3 = 4c^2s^2 - 4cs^2(b+a)$; whence $s = \frac{1}{2}\left\{c - (b+a)\right\}$, and by means of this

$$x = \frac{(s^2-ab)^2}{4s^2(a+s)(b+s)} = \frac{\left\{\left[c-(a+b)\right]^2 - 4ab\right\}^2}{8\left\{c-(a+b)\right\} \times (c+a-b) \times (c+b-a)}.$$

PROB. VI. *By Mr.* JAMES CUNLIFFE.

To find three square numbers such, that their sum being severally added to their three sides shall make square numbers.

SOLUTION.

In the first place it may be observed that 4, 9, and 36 are three square numbers, whose sum is a square, viz. 49; wherefore let the required squares be denoted by $4x^2$, $9x^2$ and $36x^2$. Now the sum of these being severally added to their three sides gives, $49x^2 + 2x$, $49x^2 + 3x$, and $49x^2 + 6x$, which expressions are all

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all to be ſquares by the queſtion:—or dividing by 49 which is a ſquare number, there will be had

$x^2 + \frac{2x}{49}$, $x^2 + \frac{3x}{49}$, and $x^2 + \frac{6x}{49}$, and theſe muſt conſequently be all ſquares.

The queſtion now adapts itſelf to the general form, in the preceding problem, therefore we may put $a = \frac{2}{49}$, $b = \frac{3}{49}$, and $c = \frac{6}{49}$; whence, and by means of what has been done,

$$x = \frac{\left\{[c-(a+b)]^2 - 4ab\right\}^2}{8\left\{c-(a+b)\right\} \times (c+a-b) \times (c+b-a)} = \frac{529}{13720};$$

and therefore the roots of the three ſquares will be

$2x = \frac{1058}{13720}$, $3x = \frac{1587}{13720}$, and $6x = \frac{3174}{13720}$.

*** This is No. 22 of Mr. Bonnycaſtle's collection of Diophantine Problems.

PROB. VII. *By Mr.* JAMES CUNLIFFE.

To find three ſquare numbers in arithmetical progreſſion, ſuch that if each be added to its reſpective root, the three ſums thence ariſing ſhall be rational ſquares.

SOLUTION.

In the firſt place $49x^2$, $25x^2$ and x^2 are three ſquares in arithmetical progreſſion; and each of them being added to its reſpective root gives

$49x^2 + 7x$, $25x^2 + 5x$, and $x^2 + x$, which three expreſſions are all to be rational ſquares by the queſtion.

Let the firſt and ſecond be divided by 49 and 25 reſpectively, and there will be had

$x^2 + \frac{x}{7}$, $x^2 + \frac{x}{5}$, and $x^2 + x$, which muſt conſequently be all ſquares.

This queſtion is now adapted to the general form in Prob. V and we may put $a = \frac{1}{7}$, $b = \frac{1}{5}$, and $c = 1$, which gives

$$x = \frac{\left\{[c-(a+b)]^2 - 4ab\right\}^2}{8\left\{c-(a+b)\right\} \times (c+a-b) \times (c+b-a)} = \frac{151321}{7863240};$$

Whence

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Whence the roots of the three squares are

$$7x = \frac{1059247}{7863240},\ 5x = \frac{756605}{7863240},\ \text{and } x = \frac{151321}{7863240}.$$

*** The preceding question is answered in the fourth number of the Student.

The following queſtion with various others of a ſimilar nature might be anſwered by a like method of procedure, if any one would be at the trouble of the calculation, *viz.*

To find three ſquare numbers ſuch that the ſum of every two ſhall be a rational ſquare; and if each be added to its reſpective root, the ſums thence ariſing ſhall be rational ſquares.

PROB. VIII. *By Mr.* JAMES CUNLIFFE.

To divide a given number (n) into four ſuch parts, that the difference of every two of them may be a ſquare number.

SOLUTION.

Let the required parts be denoted by x, y, z, and v,

Then by the queſtion $\left\{\begin{array}{l} x+y+z+v=n \\ x-y=c^2 \\ x-z=b^2 \text{ and} \\ x-v=a^2 \end{array}\right\}$ alſo $z-y$, $z-v$, and $y-v$ muſt be ſquares.

One-fourth of the ſum of the four firſt gives

$x = \frac{1}{4}(n+a^2+b^2+c^2)$, and from the ſecond
$y = x - c^2 = \frac{1}{4}(n+a^2+b^2-3c^2)$, and from the third
$z = x - b^2 = \frac{1}{4}(n+a^2+c^2-3b^2)$, and from the fourth
$v = x - a^2 = \frac{1}{4}(n+b^2+c^2-3a^2)$. And by means of theſe

$z - y = b^2 - c^2 =$ a ſquare
$z - v = a^2 - b^2 =$ a ſquare, and
$y - v = d^2 - e^2 =$ a ſquare.

From whence it appears that the relation of the three ſquares a^2, b^2, and c^2 muſt be ſuch, as will make the difference of every two of them a rational ſquare; a^2 being the greateſt, and c^2 the leaſt.

Now in one of the ſolutions to Prob. 1, it was found that a, b, and c may be reſpectively denoted by 697, 185, and 153, or we may put

$a^2 = 485809$, $b^2 = 34225$, and $c^2 = 23409$; whence, and by means of what is done above,

$x =$

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$x = \frac{1}{4}(n + a^2 + b^2 + c^2)$, $y = x - c^2 = \frac{1}{4}(n + 449807)$, $z = x - b^2 = \frac{1}{4}(n + 406543)$ and $v = x - a^2 = \frac{1}{4}(n - 1399793)$; where n may be taken at pleaſure.

But the leaſt value of n ſo as to give x, y, z, and v, all poſitive integers will be readily found to be 1399797, and from thence

$x = 485810$, $y = 462401$, $z = 451585$, and $v = 1$.

It is eaſy to perceive, that as (n) may denote any number whatever, we may from what is done, find four numbers ſuch that the difference of every two of them may be a ſquare, and the ſum of all four a ſquare, cube, biquadrate, &c.

PROB. IX. *By Mr.* JAMES CUNLIFFE.

To find four numbers ſuch, that the ſum of every two of them may be a rational ſquare; and alſo that the ſum of all four may be a rational ſquare.

SOLUTION.

Let x, y, z, and v denote the numbers.

Then by the queſtion
$$\begin{aligned} x + y &= a^2 \\ x + z &= b^2 \\ x + v &= c^2 \\ y + v &= d^2 \\ y + z &= \text{a ſquare, and} \\ z + v &= \text{a ſquare.} \end{aligned}$$

The ſum of the firſt and third is $2x + y + v = a^2 + c^2$;

Half the diff. between this and the fourth is $x = \frac{1}{2}(a^2 + c^2 - d^2)$;

Whence $y = a^2 - x = \frac{1}{2}(a^2 + d^2 - c^2)$

$$z = b^2 - x = \frac{1}{2}\left\{2b^2 + d^2 - (a^2 + c^2)\right\}$$

and $v = d^2 - y = \frac{1}{2}(d^2 + c^2 - a^2)$.

By means of the preceding expreſſions,

$y + z = b^2 + d^2 - c^2 =$ a ſquare $= e^2$; and

$z + v = b^2 + d^2 - a^2 =$ a ſquare $= f^2$;

Theſe two laſt give $b^2 + d^2 = c^2 + e^2 = a^2 + f^2$.

From the firſt part of the ſolution to the firſt problem it appears that we may put

$$c = \frac{2rsb - (r^2 - s^2)d}{r^2 + s^2}, \text{ and } a = \frac{(p^2 - q^2)b - 2pqd}{p^2 + q^2},$$

where r, s, p, and q may be expounded by any numbers at pleaſure.

The ſum of the ſecond and fourth equations gives

$x + y + z + v = b^2 + d^2 =$ a ſquare by the queſtion,

Therefore

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Therefore we may put $2mn = b$, and $m^2 - n^2 = d$,

Whence $c = \frac{2rsb - (r^2 - s^2)d}{r^2 + s^2} = \frac{4rsmn - (r^2 - s^2)(m^2 - n^2)}{r^2 + s^2}$,

and $a = \frac{(p^2 - q^2)b - 2pqd}{p^2 + q^2} = \frac{(p^2 - q^2)2mn - 2pq(m^2 - n^2)}{p^2 + q^2}$,

and m and n may be taken at pleasure.

Example. If $m = 2$, $n = 1$, $r = 3$, $s = 2$, $p = 4$, and $q = 1$, then $b = 2mn = 4$, $d = m^2 - n^2 = 3$,

$$c = \frac{4rsmn - (r^2 - s^2)(m^2 - n^2)}{r^2 + s^2} = \frac{33}{13},$$

and $$a = \frac{(p^2 - q^2)2mn - 2pq(m^2 - n^2)}{p^2 + q^2} = \frac{36}{17}.$$

And reducing these values of b, d, c and a to a common denominator, and omitting it, we may put $b = 4 \times 13 \times 17 = 884$, $d = 3 \times 13 \times 17 = 663$, $c = 33 \times 17 = 561$, and $a = 36 \times 13 = 468$. Or, $a^2 = 219024$, $b^2 = 781456$, $c^2 = 314721$, and $d^2 = 439569$, and by means of these $x = \frac{1}{2}(a^2 + c^2 - d^2) = 47088$, $y = a^2 - x = 171936$, $z = b^2 - x = 734368$, and $v = d^2 - y = 267633$, which are four numbers that will answer the question.

By attending to what has been deduced in the foregoing solution, it will appear that any number whatever, which is capable of being divided into two rational squares, may be divided into four such parts, that the sum of every two of them may be a rational square. And this property of being capable of being divided into two rational squares is a restriction that seems indispensibly necessary in the number to be divided, as specified above.

ARTICLE XIV.

Answers to the Mathematical Questions proposed in ARTICLE LI. VOL. II.

I. QUESTION 241, *answered by Mr.* G. Buffham, *of Clifton.*

Put $4x$ and x for the diameters of each vessel; $c = 10$ ale gallons $= 1{\cdot}631944$ cubic feet; $a = (8000 \times 5280 + 12) = 42240012$

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42240012 feet; and $p = \cdot 5236$. Since the surface of the liquor muſt be parallel to the ſurface of the earth, we have $4x^2 \div a$ and $x^2 \div 4a$ for the heights of two ſpherical ſegments, the diameters of whoſe baſes are the ſame as thoſe of the veſſels, the contents of which are eaſily found to be

$$(\frac{64}{a^3}x^6 + \frac{48}{a}x^4) \times p \text{ and } (\frac{1}{64a^3}x^6 + \frac{3}{16a}x^4) \times p.$$

Then by the queſtion, we have the following equation

$$(\frac{64}{a^3}x^6 + \frac{48}{a}x^4) \times p - (\frac{1}{64a^3}x^6 + \frac{3}{16a}x^4) \times p = c, \text{ or}$$

$$x^6 + \frac{68}{91}a^3x^4 = \frac{64c}{4095p}, \text{ or in numbers}$$

$$x^6 + 1333262260831756x^4 = 3671164938496296232664.$$

From this equation x is found to be equal to 40·7375, the leſs diameter, and $4x = 162\cdot95$ the greater diameter. Whence the heights of the ſpherical ſegments are ·00015715 and and ·00001 nearly, and their contents 1·6386 and ·006517 reſpectively, which being added to 109486·08585 the content of either veſſel give 109487·724456 and 109486·092367 cubic feet reſpectively, the quantity of liquor in the veſſel when full.

The ſame, anſwered by Mr. John Whitley, *of Attercliffe Academy, near Sheffield.*

Plate 29, Fig. 565. Let PFBGAEP repreſent the earth, AB, EF the diameters of the veſſels parallel to each other, and interſecting a diameter PG in the points m and n; then it is evident that the given difference will be repreſented by the zone AEFB, which is known to be equal to $(\text{En}^2 + \text{Am}^2 + \frac{\text{nm}^2}{3}) \times \frac{1}{2}\text{mn} \times p$, where $p = 3\cdot14159$ &c; therefore putting 10 ale gallons = 2820 cubic inches = 1·631944 cubic feet = a, PG = d, nm = x, En = y, and Am = z, we ſhall have this equation, *viz.* $(y^2 + z^2 + \frac{x^2}{3}) \times \frac{1}{2}px = a$. Now it is very evident by the nature of the queſtion that x will be an exceeding ſmall part, and $\frac{x^3}{3}$ will be conſiderably much more ſo, therefore the rejecting of $x^3 \times \frac{p}{6}$ will very little affect the value of x, which being done our equation becomes

(3*

$(y^2 + z^2) \times \frac{1}{2}px = a$, or $y^2 + z^2 = \frac{2a}{px}$: but by the queſt.

$4 : 1 :: z : y$, or $4^2 : 1^2 :: z^2 : y^2$, and by compoſition

$17 : 1 :: z^2 + y^2 = \frac{2a}{px} : y^2 = \frac{2a}{17px}$; conſequently

$z^2 = \frac{32a}{17px}$. Alſo by a known property of the circle,

$Pn \times Gn = y^2$, and $Pm \times Gm = z^2$; hence

$Gm - Pm = \sqrt{(d^2 - \frac{128a}{17px})}$, and $Gn - Pn = \sqrt{(d^2 - \frac{8a}{17px^2})}$,

and the difference of theſe is $2mn$, or $2x$; that is,

$$\sqrt{(d^2 - \frac{8a}{17px})} - \sqrt{(d^2 - \frac{128a}{17px})} = 2x.$$

Hence $x = 0001473$ and the greater diameter $= 162{\cdot}94624$ feet and the leſſer $40{\cdot}73656$ feet; hence the contents are eaſily found.

II. QUESTION 242, *anſwered by Mr.* Cunliffe.

Plate 29, Fig. 566. Let ABC repreſent the cone, HEGF one of the hyperbolic ſections parallel to the axis CD. Draw LE ‖ to AB and cutting CD in M; then will CM and ME be reſpectively equal to the principal ſemi-tranſverſe and ſemi-conjugate axes of the hyperbola HEG, as appears from the writers on conics.

Now by the queſtion, the double ordinate GH will be equal to the ſide of a ſquare inſcribed in the circle AHBG; and therefore

$ME = DF = GF = AB \div 2\sqrt{2} = DB \div \sqrt{2}$.

Now by reaſon of the parallels DE, FE;

$DB : ME = DB \div \sqrt{2} :: \sqrt{2} : 1 :: CD : CM = CD \div \sqrt{2}$; and by art. 129. Vol. I. Simpſon's Fluxions, the area of the hyperbolic ſpace HEG bounded by the double ordinate GH is expreſſed by

$$CD \times FG - CM \times ME \times \text{h. l.} \frac{CD + \frac{CM \times FG}{ME}}{CM} =$$

$$\frac{CD \times DB}{\sqrt{2}} - \frac{CD \times DB}{2} \times \text{h. l.} (\sqrt{2} + 1) =$$

CD

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$$\frac{CD \times DB\sqrt{2}}{2} - \frac{CD \times DB}{2} \times \text{h. l.}\,(\sqrt{2}+1) = CD \times DB \times .26642.$$

Again, by art. 157. Vol. I. Simpſon's Fluxions, the ſolid content of the ungula HEGB is expreſſed by

$$HBG \times \frac{CD}{3} - HEG \times \frac{DF}{3} = HBG \times \frac{CD}{3} - HEG \times \frac{DB}{3\sqrt{2}};$$

and four times this being taken from the whole content of the cone leaves

$$\tfrac{2}{3} \times (DB)^3 \times DC + HEG \times \frac{2DB\sqrt{2}}{3} = DB^3 \times DC \times .9178498\frac{4}{5}$$

for the content of what remains after the four ſlices are cut off.

In the preſent caſe where $DC = 7\frac{1}{2}$ feet and $DB = \frac{13}{12}$, the ſolid content $DB^2 \times DC \times .91784984 = 8.07899$ cubic feet.

The area of the convex ſurface of the conical ungula HEGB is expreſſed by (CB ÷ DB) × area of the circular ſegment HBG as appears by art. 166, Vol. I, Simpſon's Fluxions: therefore (4CB ÷ DB) × area of the ſegment HBG will expreſs the convex part of the conical ſurface taken off by the four ſlices.

Now the whole convex ſurface of the cone is expreſſed by (CB ÷ DB) × area of the circle AHBG; from whence deducting the part cut off by the four ſlices, and there will manifeſtly remain (CB ÷ DB) × area of the ſquare inſcribed in the circle AHBG = (CB ÷ DB) × $2DB^2$ = 2CB × DB.

From whence it appears that what remains of the convex ſurface of the cone, after the four ſlices are cut off is geometrically quadrable.

Wherefore if to what remains of the convex ſurface of the cone after the four ſlices are taken off there is added the area of the ſquare baſe, $2DB^2$, together with the area of the four hyperbolic ſpaces, $\frac{4}{3}$ × HEG, there will be had

$2CB \times DB + 2DB^2 + 4DB \times CD \times .26642 = 27.4245189$ ſquare feet for the whole ſuperfices of what remains after the ſlices are taken off.

From what has been deduced we may gather the following curious

THEOREM.

If any right cone is cut by planes parallel to the axis ſo as juſt to reduce the baſe to a plane figure bounded by right lines; then the area of the remainder of the convex ſurface of the cone will be

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geometrically quadrable, its area being expreſſed by (CB ÷ DB) × the area of the ſaid reduced baſe.

The ſame, anſwered by Mr. Buffham, *Cliſton, near Newark.*

The content of the whole cone is eaſily found = 15927·912 inches.

Now the content of the piece mentioned in the queſtion is equal to the content of the whole cone, minus four times the content of the ungula HEBG.

By a property of the circle and ſquare $1 : \frac{1}{2}\sqrt{2} :: 26 : 13\sqrt{2}$ = GH the ſide of the ſquare or baſe of the ſolid = LE; and FB = ½ (AB — LE) = 3·8076118: and by ſim. Δs,
AB = 26 : DC = 90 :: LE = $13\sqrt{2}$: CM = $45\sqrt{2}$;
alſo EF = $90 - 45\sqrt{2} = 26·3604$.

Since EFB is a right angle, it is evident, from Dr. Hutton's Menſuration, Cor. 6. Prob. XXVI. Sect. 1, Part III, that the tranſverſe and conjugate axes are 127·27927 and 18·384776 reſpectively, alſo the area of hyperbolic ſection GEH = 311·71178, by Prob. VI. Sect. 7, Part III, of the ſame book, which call A; alſo denote the area of the circular ſegment GBH by *a*.

Then by Page 224 of Hutton ut ſupra,

$$\frac{\frac{1}{3}h}{D - d} \times \left(a \times D - A \times d \times \frac{DB}{CD}\right) = 491·836326,$$ the

content of the ſlice GEBH, which being multiplied by 4 and the product taken from the whole cone, or 15927·912, leaves 13960·5667 inches, or 8·079 feet nearly, the content of the part remaining after the 4 ſlices are taken off

Again, the ſuperficies of the whole cone = 26 × 3·1416 × 90·934 ÷ 2 = 3713·817297, and by Prob. XXXII. Sect. 1, Part III, of the above mentioned book, the curve ſurface of one

$$\text{ſlice} = \sqrt{\left\{(4h + D - d) \div (D - d)\right\}} \times a = 337·3794;$$

this multiplied by 4 gives 1349·5176, which taken from the ſurface of the whole cone leaves 2364·29969; to this adding four times the area of the hyperbolic ſection GEH (found above) = 1246·84712 and the area of the ſquare baſe = $(13\sqrt{2})^2 = 338$, we have 3949·1465 inches, or 27·4246 ſquare feet for the ſuperfices of the part remaing after the four ſlices are taken taken off.

Neat and ingenious ſolutions were alſo received from Meſſrs. Francis, Marrat, *and* Whitley.

III.

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III. QUESTION 243, *answered by Miss* Susan May.

Plate 29, Fig. 567. Let ACB be the triangle required; AD, BE, and CF the ⊥s demitted from the angles on the opposite sides, and which meet in a point, as at G. Join DE, DF, and EF. Then because the points B, F, G, D are in a circle, and also the points C, D, G, E, we have ∠GBF = GDF, and ∠ECG = EDG: but ∠GBF = GCE, therefore EDG = FDG; and in like manner it may be shewn that DEG = GEF and GFE = GFD.

Now in the △ DEF the three sides are given, *viz.* EF = 9, ED = 10, and FD = 11; therefore the three ∠s are found by trigonometry, *viz.* DFE = 58° 59′ 42″, EDF = 50° 28′ 50″, and DEF = 70° 31′ 28″; therefore ∠ ADF = 25° 14′ 25″; ∠ DFA = 119° 29′ 52″, and DAF = 35° 15′ 45″; hence AD is found = 16·585, BD = 11·724, and CD = 9·386. Consequently (CD + DB) × ½AD = 21·11 × $\frac{16\cdot585}{2}$ = 175·04 square chains = 17½ acres, the content required.

The same, answered by Tyro Philomatheticus, *of Hull.*

Plate 29, Fig. 568. Let ABC represent the triangular field, AD, BE, and CF the ⊥s from the ∠s on the opposite sides, G the point of intersection, and ED, EF, FD the given lines. Before I proceed to the calculation it may be thought necessary to prove (as I have not yet seen it done) that, *The perpendiculars from the angles on the opposite sides bisect the angles of the triangle formed by joining the extremities of those perpendiculars.*—On AB describe a semicircle, then because of the right ∠s AEB and ADB, it will pass through the points E and D; and AEG, AFG being right angles it is manifest that the points A, F, G, and E are in the circumference of a circle, and the ∠ EFG = EAG; again BFG, BDG, being right ∠s, it is likewise evident that B, F, G, and D are in the circumference of a circle, and ∠DFG = DBG: but the ∠s EAG and DBG, standing on the same segment ED, are equal; therefore EFG = DFG; and in a similar manner it is easily proved that FDG = EDG, and that DEG = FEG.

Now ED = 9, EF = 10, and FD = 11, whence the ∠s by Trig. are found, *viz.* FED = 70° 32′, EDF = 58° 59½′ and EFD = 50° 28½′ from which, and what is done above, we get ABC (= comp. of ½ FED) = 54° 44′, CAB (= comp.

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of $\frac{1}{2}$ FDE) $=$ 60° 30$\frac{1}{2}$′, and BCA ($=$ comp. of $\frac{1}{2}$EFD) $=$ 64° 45$\frac{1}{2}$′; and by trigonometry, as
fine of $\angle$ A : EF :: cos. $\frac{1}{2}$FED : AF $=$ 9·3808, and
fine of $\angle$ B : FD :: cos. $\frac{1}{2}$EDF : FB $=$ 11·7264;
therefore AB $=$ 21·1069; again as
fine of $\angle$ A : EF :: cos. $\frac{1}{2}$EFD : AE $=$ 10·3925, and
fine of $\angle$ C : ED :: cos. $\frac{1}{2}$EDF : EC $=$ 8·66015;
therefore AC $=$ 19·05265; hence the area $=$
AB $\times$ AC $\times$ $\frac{1}{2}$ fine $\angle$A $=$ 175·01 chains nearly, or 17$\frac{1}{2}$ acres.

The same, answered by Mr. John Whitley.

Plate 29, Fig. 569. Let ABC represent the piece of ground: from the $\angle$s, to the opposite sides, draw the $\perp$s AE, BF, CD; draw also the lines DF, DE, EF, intersecting the $\perp$s in *n*, *p*, *m*, respectively. Since AEB and AFB are right $\angle$s, the points A, F, E, B are in a circle, and for the same reason the points C, E, D, A are in a circle. Therefore $\angle$ ABF $=$ AEF, and ACD$=$AEB: but $\angle$ABF$=$ACD, because each of these is the complement of the same $\angle$A; therefore $\angle$AEF $=$ AED, and consequently the $\perp$ AE bisects the $\angle$FED. In the same manner it may be proved that the $\angle$s EDF, and DFE are respectively bisected by the $\perp$s CD and BF.

Having premised this the calculation may be as follows.
Put DE $=$ 9, FD $=$ 11 and EF $=$ 10; then by Euc. 3, VI,
FE $+$ ED : FD :: FE : F*n* $=$ 110 $\div$ 19, whence
D*n*$=$FD$-$F*n*$=$11$-$110$\div$19$=$99$\div$19, and by Em. Geo. 26, II,
E*n*^{2} $=$ FE$\times$ ED$-$F*n*$\times$ L*n*$=$21600$\div$19^2, or E*n*$=$60$\sqrt{6}\div$19.
Also FE $+$ F*n* : E*n* :: FE : EO $=$ 2$\sqrt{6}$, whence
*n*O $=$ E*n* $-$ EO $=$ 22$\sqrt{6}\div$ 19.
Because ADO, AFO are right $\angle$s, the points A, F, O, D are in a circle; therefore, Euc. 35, III,

$$A*n* \times *n*O = F*n* \times *n*D, \text{ or } A*n* = \frac{F*n* \times *n*D}{*n*O} = \frac{5 \times 33\sqrt{6}}{2 \times 19}, \&c.$$

$$AO = A*n* + *n*O = \frac{5 \times 33\sqrt{6}}{2 \times 19} + \frac{22\sqrt{6}}{19} = \frac{11\sqrt{6}}{2}.$$ Again

DF $+$ FE : DE :: DF : D*p* $=$ 33$\div$7, whence *p*E$=$30$\div$7; & F*p*^{2}$=$DF$\times$ FE$-$D*p*$\times$ *p*E$=$440$\div$49, or F*p*$=$20$\sqrt{11}\div$7. Also FE$+$E*p* : F*p* :: FE : FO$=$2$\sqrt{11}$, and by the sim. $\triangle$s, FOE, AOB;
FO : FE :: AO : AB $=$ 5$\sqrt{66}\div$ 2. Again
FD $+$ DE : FE :: FD : F*m* $=$ 11 $\div$ 2, whence *m*E $=$ 9 $\div$ 2.

Wherefore

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Wherefore $Dm^2 = FD \times DE - Fm \times mE = \frac{3}{4} \times \frac{90}{1} = \frac{9}{4} \times 33'$

or $Dm = \frac{3}{2}\sqrt{33}$. Alſo

$DE + Em : Dm :: DE : DO = \sqrt{33}$; whence $Om = \frac{1}{2}\sqrt{33}$.

And $Cm \times mO = Fm \times mE$, or $Cm = \frac{Fm \times mE}{mO} = \frac{3}{2}\sqrt{33}$.

Therefore $Cm = Dm$ and $DC = 3\sqrt{33}$. Whence the area is

$\frac{AB \times DC}{2} = \frac{5}{2}\sqrt{33} \times \frac{3}{2}\sqrt{66} = \frac{15}{4} \times 33\sqrt{2} = \frac{15}{4} \times 33 \times$

1·41421356 = 175·008928 ſquare chains = 17·5 acres, near'y.

And thus the queſtion was anſwered by Mr. Cunliffe;

Who obſerves "What is deduced previous to the calculation ſuggeſts an eaſy way of conſtructing the problem geometrically, and which may be as follows, *viz.*—With the given diſtances of the points, where the perpendiculars from the angles meet the oppoſite ſides, let the triangle FDE be formed, and having biſected the angles FDE, FED, and DFE with the lines *Dm*, E*n*, and F*p*, draw the lines AB, BC, and AC reſpectively perpendicular to the former, through the points D, E, and F, and they will form the triangle required".

Solutions were alſo received from Meſſrs. Francis, Lowry, *and* Swale.

IV. QUESTION 244, *anſwered by Mr.* Cunliffe.

Plate 29, Fig. 570. Let the triangle FAE repreſent a vertical ſection of the wall at right angles to the face next to the water, and let the point S in AE denote the ſurface of the water. Draw FP to any point P*m* in the line AS. By the principles of Hydroſtatics, the preſſure of the water at P againſt AE will be expreſſed by $m \times SP$, where *m* denotes the ſpecific gravity of the water; and by the reſolution of Forces, its force in a direction perpendicular to FP will be expreſſed by $m \times SP \times (PA \div FP)$: and therefore the effect of the preſſure of the water againſt the point P to turn the triangle FAE about F will be expreſſed by $m \times SP \times (PA \div FP) \times FP = m \times SP \times PA$.

Conſequently the effect of the preſſure of the water againſt the whole line AS, to turn the triangle FAE about F will be expreſſed by the ſum of all the $m \times SP \times PA$.

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Now to find the sum of all the $m \times SP \times PA$, put $AS = d$ $SP = x$, then $PA = d - x$; and the sum of all the $m \times SP \times PA = \text{flu. } m\,x\dot{x}\,(d-x)$ when $x = d$: but $\text{flu. } mx\dot{x}\,(d-x) = \text{flu. } md\dot{x}x - \text{flu. } m\dot{x}x^2$

$$= m \times \left(\frac{dx^2}{2} - \frac{x}{3}\right),$$

and when $x = d$, the fluent becomes

$m \times \left(\frac{d^3}{2} - \frac{d^3}{3}\right) = \frac{md^3}{6} = \frac{n}{6} \times AS^3$, which as is shewn above expresses the effect of the pressure of the water upon the line AS to turn the triangle FAE about F.

To find the effect of the weight of the triangle FAE to turn it in a contrary direction. From G, the centre of gravity of the triangle, draw GN perpendicular to FA, and let W denote the weight of the triangle. Then by the principles of mechanics, the effect of the weight of the triangle FAE to turn it about F in a direction contrary to that of the action of the water will be expressed by

$$W \times FN = \tfrac{1}{2} \times FA \times AE \times n \times FN$$
$$= \tfrac{1}{6}n \times FA \times AE \times FN$$
$$= \tfrac{1}{3}n \times FA^2 \times AE;$$

because $W = \frac{1}{6}n \times FA \times AE$ and $FN = \frac{2}{3} \times FA$; where n denotes the specific gravity of the wall.

Therefore when the wall just supports the fluid, the two preceding forces must be equal to each other, *viz.*
$\frac{1}{3}n \times FA^2 \times AE = \frac{1}{6}m \times AS^3$ whence

$$FA^2 = AS^2 \times \frac{m \times AS}{2n \times AE}, \text{ or } FA = AS\sqrt{\frac{m \times AS}{2n \times AE}}.$$

In the present case where $AS = 10$, $AE = 12$, $m = 7$, and $n = 11$,

$$FA = AS\sqrt{\frac{m \times AS}{2n \times AE}} = 10\sqrt{\frac{35}{132}} = 5{\cdot}149286 = 5\,\frac{3}{20}$$

feet nearly, the thickness of the wall at the bottom, just sufficient to support the fluid.

Messrs. Buffham *and* Marrat *also sent answers to this question.*

V. QUESTION 245, *answered by Mr.* Gregory, *the Proposer.*

Plate 29, Fig. 571. It has been demonstrated that the altitude AK of the greatest inscribed triangle is equal to $\frac{3}{4}$ of the transverse diameter; it has also been demonstrated that the altitude LB of the least circumscribing triangle is $\frac{3}{2}$ of the transverse: And it

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it will readily appear, since LF is half of LB, that EF = HK is = ¼BM. Therefore, the triangles LMN and AHI are similar, and as the altitude of the former is double that of the latter, its area is quadruple; and the difference between the areas is equal to thrice AHI. Hence 70·148025 the given difference divided by 3 gives 23·382675 area of AHI. Let this be divided by 9 = AK = ¾AB, the quotient is 2·598075 = ⅓√3 = HK.

Then, by the nature of the ellipse,

$AK \times KB : HK^2 :: AG \times GB : GC^2$;

that is as $27 : \frac{27}{4} :: 36 : \frac{36 \times 27}{27 \times 4} = 9 = GC^2$.

Consequently GC = 3 and CD = 6 the conjugate required.

The same, answered by Miss Susan May.

Plate 29, Fig. 572. Let BEAF represent the ellipsis required, QPR the least circumscribing triangle, and EBF the greatest inscribed one.

Now it is manifest from the first solution to quest. 170 of Repository, that

GP = GB = ¾BA = 9, and

$\triangle QPR = 27 \times GE\,(BR \times BP) = 4 \triangle EFB\,(GE \times GB)$; therefore $3GE \times GB = 70{\cdot}148025$, or $GE = 70{\cdot}148025 \div 27 = 2{\cdot}598075$.

And by the property of the ellipse,

$(\text{conjugate})^2 : 12^2 :: GE^2 : BG \times GA = \frac{3}{16}(12)^2$, that is,

the conjugate $= 4GE \div \sqrt{3} = 10{\cdot}392\,3 \div 1{\cdot}732\,05 = 6$.

The same, answered by Mr. William Marrat, *Boston.*

Plate 29. Fig. 572. Let AFB be half the eliptical table, and draw the lines as per figure. The greatest inscribed $\triangle BFG$ is when the altitude BG is ⅔ of the transverse axe AB; and the least circumscribing $\triangle$ (by Prob. 5, Emerson on Curve Lines) is, when the subtangent PG = GB; and by Prop. IX, Ellipse, Hutton's Conics,

PB : PO :: PG : PA :

and alternately PB : PG :: PO : PA;

but PB = 2PG, therefore PO = 2PA, and conseq. AG = GO.

Now GB = ¾AB = 9, and PB = 18, therefore putting GF = x, twice the area of the $\triangle$ BFG = $9x$, and twice that of the $\triangle$ PBR = $36x$; therefore by the question $36x - 9x = 70{\cdot}148025$; hence $x = 2.598075$.

Then,

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Then, by the property of the ellipſe,
$AG \times GB : GF^2 :: AB^2 : CD^2 = 36$.
Conſequently $CD = 6$, the conjugate diameter required.

The ſame, anſwered by Mr. G. Buffham.

Plate 29, Fig 572. Let the ellipſe AEBF repreſent the table, O its centre, and let lines be drawn as in the figure. Put t and c for the tranſverſe and conjugate diameters reſpectively; put alſo $BG = x$. Then, by conics, $GE = (c \div t) \sqrt{(tx - x^2)}$, and the area of the $\triangle$ $BEF = (c \div t)\, x \sqrt{(tx - x^2)}$, which by the queſtion muſt be a maximum, or $tx^3 - x^4$ a maximum, which being fluxed and reduced gives $x = \frac{3}{4}t = 9$; and, by ſubſtitution, $(c \div t)\, x\sqrt{(tx - x^2)}$, becomes $\frac{3}{16}tc\sqrt{3}$.—By the Scholium to Prob. 8. Simpſon on the maxima et minima of geometrical quantities, tangents drawn to the ellipſe through the points E and F form the leaſt circumſcribing $\triangle$; for, by the ſaid ſcholium, it is a minimum when the ſubtangent $GP = BG$; and per conics,
$OG : GA :: BG : GP$; therefore $OG = GA$, or
$BG\ (x) = \frac{1}{4}AB\ (\frac{1}{4}t)$ as before

Whence the area of the triangle PQR is eaſily found to be $(4c \div t)\, x\sqrt{(tx - x^2)} = 3c\sqrt{27}$, by ſubſtituting for x and t. Then per queſtion $3c\sqrt{27} - (3 \div 16)\, tc\sqrt{3} = 70{\cdot}148025$. Hence $c = 6$ the conjugate required.

The ſame again, by Mr. John Whitley.

Plate 29, Fig. 572. CONSTRUCTION. Let ABCD repreſent the elliptic table, AB the given tranſverſe, and CD the required conjugate diameter, and SLMN a rectangle equal in area to the third part of the given difference of the triangles. Upon AB, as a diameter, deſcribe the circle AIBH, and biſect its radius AO in G, and draw IGH $\perp$ to AB, meeting the circle in I and H; in IG take the point F ſo that $BG : MN :: ML : GF$, make OC (= OD) perpendicular to AB and ſuch that $IG : FG :: AO : OD$, ſo ſhall CD be the required conjugate

DEMONSTRATION. Join IB, HB, and in BG, FG, produced take PG and GE reſpectively equal to BG and FG. Join alſo PF, PE, which produce to meet RBQ drawn parallel to FE in Q and R, alſo join BF, BE. Then by *conſtruction* $AG = GO$, therefore the $\triangle$ BHI is equilateral, and is, therefore, the greateſt that can be inſcribed in the circle; and becauſe $PG = GB$, PE

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PE = EQ = BE, therefore the △s PFE, BFE, BEQ, and BFR are equal in all reſpects. But by the ſcholium to Theorem 8, Simpſon's Geometry, on the max et min. of quantities, the △ BFE is half of the greateſt parallelogram that can be inſcribed in the triangle PQR; it alſo follows from the ſaid ſcholium that the triangle PBR is the leaſt that can be circumſcribed about the curve ACB, therefore the △ BFE which is double of the △ BFG is a maximum, alſo the △ PQR which is double the △ PBR is a minimum. And by a property of the ellipſe,
AG × GB = IG^2 : FG^2 :: AB^2 : CD^2, wherefore
IG : FG :: AB : CD; but by conſtruction
BG : NM :: ML : GF, therefore, Euc. 16, VI,
BG × FG = SL × LM = the area of the triangle FBE (per *conſtr.*)
= ¼ PQR = ⅓ PQR − ¼ BFE. *Q. E. D.*

The ſame otherwiſe, by Mr. Cunliffe, *of Bolton.*

It is well known that an equilateral △ is the greateſt that can be inſcribed within, and the leaſt that can be circumſcribed about any given circle. Therefore if a circle with its greateſt inſcribed, and leaſt circumſcribing △s, be imagined to be projected *orthographically* upon a plane which makes an ∠ with the plane of the figure, the projection will be an ellipſis with its greateſt inſcribed and leaſt circumſcribing △s. And the difference between the greateſt inſcribed and leaſt circumſcribing △s in the circle will be to the difference between the greateſt inſcribed and leaſt circumſcribing △s in the ellipſis, as the tranſverſe to the conjugate diameter of the ellipſis. This is well known to thoſe acquainted with the principles of the orthographic projection.

This being premiſed, let d denote the tranſverſe, and c the conjugate diameters of the ellipſis; then it is well known that $\frac{1}{4}d\sqrt{3}$ will expreſs the ſide of an equilateral △ inſcribed in the circle whoſe diameter is d; $(3 \div 16)\, d^2\sqrt{3}$ its area. Alſo $d\sqrt{3}$ will expreſs the ſide of the leaſt circumſcribing equilateral △, which is manifeſtly double the ſide of the inſcribed △; therefore its area will be four times the area of the inſcribed △, and conſequently the difference between the areas of the ſaid △s will be equal to thrice the area of the inſcribed △ or $(9 \div 16)\, a^2\sqrt{3}$. Now let A denote the difference of the areas of the greateſt inſcribed and leaſt circumſcribing △s in the ellipſis;
then $d : c :: (9 \div 16)d^2\sqrt{3} : A$; whence $(9 \div 16)d^2c\sqrt{3} = dA$,
or $c = 16A \div 9d\sqrt{3} = (16A \div 27d) \times \sqrt{3} = 6$ feet.

VI. QUESTION 246, *anſwered by Mr.* Boſworth, *the Propoſer.*

In Fig. 573, Pl. 29. Let AB and AC repreſent the two mirrors, and a the ſmall object on AB, placed 6 inches from B. It has

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has been proved that the distance *bc* of the image formed by reflection from AC, is equal to the distance of the object *a* from that mirror; and that the images *l*, *g*, *c*, *e*, *i*, formed by continual reflections from both mirrors are in the circumference of a circle whose centre is A and whose radius is the distance of the object *a* from that centre. See *Enfields Institutes of Nat. Phil.* Prop. CI, and CII; and *Wood's Optics*, Prop. XIII. XIV. From the known principles of optics we are certain that *ac*, *eg*, and *il*, are perpendiculars to AC, and bisected by it; in like manner *ce*, *gi*, and *lm*, are perpendiculars to AB, by which they are bisected. It follows, then that the ∠s *ace*, *ceg*, *egi*, *gil*, *ilm*, are each = the given ∠ CAB = 15°; and, by Euc. III. 20. the arcs *lg*, *gc*, *ca*, *ae*, *ei*, *im*, each = 30°—whence the number of images formed on the two mirrors = 180 ÷ 30 = 6; but the last is not perfect, as the centre of it falls on A, the point of interfection, and therefore only half of it is caught by the mirror. If twice the ∠ of inclination be not a measure of 180°,—the number of images formed will be that whole number which is next inferior to the quotient of 180° by twice the angle.

To find the distance of each image behind the surface of the mirror upon which it is formed, we must consider that the first distance is denoted by the fine of the same number of degrees as the inclination of the mirrors; the second is the fine of twice that number; the third of three times, and so on—for *bc* is, evidently, the fine of 15°, in the present case; *de* of 30°—and so of the rest. These distances may easily be calculated by a table of natural fines; thus,

			In.	Inches.
As 1 (Rad) :	·2588190 = fin. of 15°	::	36 (Rad) :	9·3172840 = *bc*
	·5000000 = ... 30°	::	36 .... :	18° ..... = *de*
	·7071068 = ... 45°	::	36 .... :	25·4556648 = *fg*
	·8660254 = ... 60°	::	36 .... :	31·1769144 = *hi*
	·9659258 = ... 75°	::	36 .... :	34·7733288 = *kl*
				36°..... = *lm*.

The same, answered by Miss Susan May.

Plate 29, Fig 574, 575. Let CA′, CB′, be two plain mirrors inclined in a given angle A′CB′, and let *a* be an object placed between them. With the distance C*a* and centre C describe a circle. Draw *a*O ⊥ to CB′ meeting the circle at O, and *a*I ⊥ to CA′ meeting the circle at I. Draw also OK ‖ to *a*I, ID ‖ to *a*O, DL ‖ to OK, KE ‖ to DI, EM ‖ to DL, FL ‖ to KE, and so on. Then because of the parallels it is evident that
DI, EK, LF, GM, HN, &c. are ⊥ to CB′, and that
OK, DL, EM, FN, GQ, &c. are ⊥ to CA′;
therefore by the property of the circle,

am =

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$am = mo$, $In = nD$, $Ko = oE$, $Lp = pF$, $Mq = qG$, $Nr = rH$, &c. and $ax = xI$, $Ow = wK$, $Dv = vL$, $Eu = uM$, $Ft = tN$, $Gs = sO$, &c.

Now it is well known that in a plain mirror the object appears to be as far behind as it is distant from the mirror, therefore O will be the place of the object *a*, as reflected from the mirror B'C, this image will be reflected again from O to K by the mirror A'C, then from K to E by the reflection of BC, and so on.

Again, I will be the place of the image of the object as reflected by A'C, this image will be reflected by BC from I to D, by AC from D to L, and so on.

Therefore O, D, E, F, G, H, &c. will be the number of images of the object reflected by the mirror B'C, and I, K, L, M, N, O, &c, those reflected by the mirror A'C. Now when the object is placed on one of the mirrors, as A'C, the images at I, D, L, F, N, &c. will be wanting, and the others will appear as in Fig. 575, where it is evident that the arcs *a*O, *a*K, OE, KM, EG, NQ, are equal, each being equal to twice the given $\angle\ aCb = 30°$; hence $180 \div 30 = 6$, the number of images formed on the two mirrors, the last of which is imperfect, as it falls on the point of intersection of the mirrors. The distances Om, wK, oE, uM, and Gq, are easily determined, for they are respectively the lines of 15°, 30°, 45°, 60°, and 75°. Hence by *Page* 26, *of Emerson's Trigonometry*—$Om = 18\sqrt{(2 - \sqrt{3})}$, $wK = 18$, $oE = 18\sqrt{2}$, $uM = 18\sqrt{3}$, and $Gq = 18\sqrt{(2 + \sqrt{3})}$.

Messrs. Cunliffe, Gregory, *and* Lowry, *also favoured us with neat solutions to this question.*

VII. QUESTION 247, *answered by Mr.* William Francis, *Teacher of the Mathematics at Maidenhead.*

Plate 29, Fig. 576. Let ABCDE be the pentangular garden, and *o* the centre of its circumscribing circle. Let lines be drawn as in the figure, and draw the radius A*o* which will be ⊥ to EB at *h*. Now it is evident that the figure *cdeab* will be a pentagon, and therefore the most regular inscribed curve will be a circle whose radius is *oh*. The $\angle\ hAE = 54°$ and $AE = 50$; hence A*h* is found = 29·390, and $hE = 40·45085$: but 10 times *h*E is = to the sum of all the walks = 404·5085. Again, A*o* is found = 42·5332, and A*o* — A*h* = *oh* = 13·1432 the radius of the circle, and its area = 542·697 yards, which at 6d. per yard amounts to 13l. 11s. 4d.

The same, answered by Mr. Gregory, *the Proposer.*

It is plain that the angles BAC, and BCA are each $= 360 \div 10 = 36°$, and hence $ABC = 180 - 72 = 108°$. Therefore,

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as S. of 36° : S. of 108° :: AB (50) : 80·90168852 = AC = AD = BD = BE = EC; the ſum of the lengths of theſe five walks is conſequently = to 80·90168852 × 5 = 404·588426 yards. Now $CaB = CAB + ABE = 36° \times 2 = 72°$, and $CBd = 180° - (CdB + ACB) = 72°$:
therefore CdB is an iſoſceles triangle, and $Cd = CB$:
in the ſame way it is ſhewn that $Ab = AB$ or CB.
Hence $Ce = AC - BC = 30·90168852$, and $de = AC - 2Ce = 19·0983148$. After a ſimilar way we might find *ae*, *ba*, *cb*, or $dc = 19·0983148$. It is manifeſt that $AdB = 108° - 36° \times 2 = 108°$; and of courſe the oppoſite angle $cde = 108°$: thus it may be ſhewn that the angles *dea*, *eab*, *abc*, and *bcd*, are each = 108°—and, conſequently the figure *abcde* having five equal ſides, and five equal angles, is a regular pentagon: and the moſt regular inſcribed curve will be a circle. Then, as S. of AoB : S. of ABo :: AB : $Ao = 42·5332$, alſo as, Rad. : AB :: S. of EBA : $Ah = 29·390$; hence $Ao - Ah = 13·1432$ the radius of the circle. The content of which is 542·697 ſquare yards; and this at 6d. per yard amounts to 13l. 11s. 4d. the expence of digging the pond.

The ſame, anſwered by Tyro Philomatheticus.

Let ABCDE repreſent the garden, &c. Then it is evident that the pond muſt be a circle, the centre of which is the centre of the pentagon; it is likewiſe manifeſt that Dko is a right angle, and that $Dk = kB$, and from the nature of the figure $oDk = 18°$ and $kCD = 54°$; therefore as, Rad. : DC (50) : : S. of kCD (54°) : $Dk = 40·45085$, and $40·45085 \times 10 = 404.5085$, the ſum of the lengths of all the walks; and as, S. of Dok (72°) : Dk (40·45085) :: S. of oDk (18°) : $ok = 13·1432$ the radius of the pond; conſequently its area is 542·697 ſquare yards, which at 6d. per yard amounts to 13l. 11s. 4d. the expence of digging.

The ſame, anſwered by Mr. John Whitley.

Let ABGDEA repreſent the pentangular garden, AD, BE, AC, DB, and CE, the walks interſecting each other in the points *b*, *c*, *d*, *e*, *a*; then, by *Emerſon's Geometry* IV. 43, $DEdC$, $DEAe$, $EABa$, $ABCb$, and $BCDc$, are parallelograms, and therefore the angles CdE, CDE, DeA, DEA, &c. are equal, and conſequently *abcdea* is a pentagon, and the curve which touches all its ſides will be a circle, the centre of which will be, the centre of that circle which circumſcribes the given pentagon. Let *o* be the centre and draw $og \perp$ to AD, and through *o* draw AoH meeting DC in H. Then, by Cor. 3. of the above quoted propoſition, we have as,

1:

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$1 : (1 + \sqrt{5}) \div 2 ::$ ſo is the ſide of the pentagon : to the diagonal, or the length of one of the walks ; that is,
As $1 : (1 + \sqrt{5}) \div 2 :: DC (= 50) : DA = 25 (1 + \sqrt{5})$;
alſo, by Prop. 44, of the ſame Book,
As $5 - \sqrt{5} : 2 :: DC^2 : oA^2 = 2 \times 2500 \div (5 - \sqrt{5}) = 1250 + 250\sqrt{5}$;
and, by ſim. $\triangle$s, $DA : DH :: oA : og$, that is,
$25(1+\sqrt{5}) : 25 :: \sqrt{(1250+250\sqrt{5})} : \sqrt{(1250+250\sqrt{5})} \div (1 + \sqrt{5}) = og$ the radius of the circle touching the walks, the area of which is eaſily found $= 542{\cdot}7$ ſquare yards nearly, which at 6d per yard comes to 13l. 11s. 4d. and the ſum of the lengths of all the walks is equal to $(1 + \sqrt{5}) \times 25 \times 5 = 404{\cdot}5085$ yards.

W. W. R.

Ingenious ſolutions were alſo received from Meſſrs. Buffham, Lowry, May, *and* Thornoby.

VIII. QUESTION 248, *anſwered by Mr.* John Lowry.

The curve required is an exponential one, whoſe equation is $x = y^y$, where x repreſents the abſciſſa, and y the correſponding ordinate.

For if Y be put for the hyperbolic logarithm of y, then $x = y \times Y$ and $\dot{x} = \dot{y} \times Y + \dot{Y} \times y$; but by the nature of logarithms $\dot{Y} = \dot{y} \div y$; then by ſubſtitution $\dot{x} = \dot{y} \times Y + \dot{y}$, therefore $\dot{y}x \div \dot{y}$ (the general expreſſion for the ſubtangent) $= y \times Y + y$: but $x = y \times Y$; therefore the ſubtangent $= x + y$, and the diſtance of the ſubtangent from the vertex is $=$ to the ordinate y.

The ſame, anſwered by Mr. Marrat, *of Boſton.*

If x denote the abſciſſa and y the ordinate of the curve required ; then by the property of tangents, $\dot{y} : \dot{x} :: y :$ the ſubtangent, that is, by the nature of the queſtion, $\dot{y} : \dot{x} :: y : y + x$, and by reduction $x\dot{y} - \dot{x}y + y\dot{y} = 0$. To find the fluent put $y = xz$, then $\dot{y} = z\dot{x} + z\dot{x}$, therefore by ſubſtitution we obtain $z^2x\dot{x} = -x^2\dot{z} - x^2z\dot{x}$; therefore $\frac{\dot{x}}{x} = -\frac{\dot{z}}{z^2} - \frac{\dot{z}}{z}$, the fluent of which is, hyp. log. $x = \frac{1}{z}$ — hyp. log. z : but $z = y \div x$; therefore hyp. log. $x = \frac{x}{y}$ — hyp. log. $\frac{y}{x}$; hence $y \times$ hyp. log. $y = x$ and conſequently $y^y = x$, the equation of the curve required.

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The same, answered by Tyro-Philomatheticus.

By the question, $\frac{y\dot{x}}{\dot{y}} = x + y$, ($x$ being the abscissa, y the ordinate, and $\frac{y\dot{x}}{\dot{y}}$ the general expression for the subtangent); therefore to find the relation of x and y, suppose $zy = x$, then by substitution we get $y(z\dot{y} + y\dot{z}) = zy\dot{y} + y\dot{y}$, or $y\dot{z} = \dot{y}$, and $\dot{z} = \frac{\dot{y}}{y}$; hence $z =$ h. l. y, and the equation of the curve is $x = y \times$ h. l. y.

The same, answered by Mr. John Whitley.

Let $x =$ the abscissa, and $y =$ the ordinate of the required curve; then by the question $y = \frac{y\dot{x}}{\dot{y}} - x$; consequently $\frac{y\dot{x} - x\dot{y}}{y^2} = \frac{\dot{y}}{y}$; the fluent of which gives $\frac{x}{y} =$ hyp. log. y; whence $x = y \times$ hyp. log. y the equation of the curve required.

This question was likewise ingeniously answered by Messrs. Cunliffe, Gregory, *and* Thornoby.

IX. QUESTION 249, *answered by Mr.* Johnson, *of Birmingham.*

For the sum of the squares.

Construction. Pl. 30, Fig. 579. Join the three given points A, B, C; from C draw the diameter CSV, and let it be divided in the given ratio at O. On OC as a diameter describe a semi-circle, and from R, the middle of AB, apply RP to cut or touch the semi-circle, as at P, and such that $2AP^2 + 2RP^2 =$ the given sum of the squares; then through P and C draw the chord CPD; join AP, BP, and it is done.

Demonstration. By Constr. CO : OV is = to the given ratio; and by Euc. 19, VI. Stone's Edition, CO : OV :: CP : PD; therefore CD is divided, at P, in the given ratio; and $2AR^2 + 2RP^2 =$ by constr. to the given sum of the squares $= AP^2 + BP^2$.

For

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For the difference of the ſquares.

Having as above joined the points A, B, C, drawn the diameter from C, and divided it in the given ratio at O, and on OC as a diameter deſcribed a ſemi-circle; then from R, the middle of AB, take RI ſuch that AB × 2RI = the given difference of the ſquares, and at I erect the perpendicular IP to cut or touch the ſemi-circle at P, through which, and C, draw the chord CPD. Join AP, BP, and it is done.

The ſame, anſwered by Mr. John Lowry.

ANALYSIS. Pl. 30, Fig. 580. Suppoſe the chord CD to be really drawn as required. Join the points A and B, and draw the diameter CS, which divide at I in the given ratio. On CI as a diameter let a circle be deſcribed, and join IP, DS. Then, becauſe of the right angles at P and D, the △s ICP, SCD are ſimilar; therefore the chord CD will always be divided in the given ratio by the circle CIP. Now when the *ſum* of the ſquares of AP, BP is given, biſect AB at H, and join HP; then $AP^2 + BP^2 = 2AH^2 + 2PH^2$; but AH is given; therefore PH is given, and H is a given point, therefore the locus of the point P is a circle whoſe centre is H and radius HP, and conſequently the interſection of this circle with the circle ICP will determine the point P.

Since AH is conſtant, the ſum of the ſquares of AP, BP, will be the greateſt or leaſt poſſible when HP is the greateſt or leaſt, that is when it paſſes through the centre of the circle ICP.

Again when the *difference* of the ſquares AP, BP is given.

Draw PK ⊥ to AB; then $AP^2 - BP^2 = AB \times 2KH$, therefore KH is given, and the locus of the point P is the ſtraight line KP given by poſition; therefore its interſection with the circle ICP will determine the point P.

The difference of the ſquares will evidently be a maximum when KP touches the circle at P.

The ſame, anſwered by Mr. Swale.

Pl. 30, Fig. 581. Suppoſe the chord CD drawn as required, and divided in the given ratio at P. Join AB and biſect it at E, and join AP, BP, and EP. Then ſince $AP^2 + PB^2 = 2PE^2 + \frac{1}{2}AB^2$ is a given ſpace, and AB a given line, it follows that PE is alſo given. Now join CE and draw DG parallel to PE meeting CE at G. By parallels PE : DG = CP : CD, a given ratio, or DG a given line: But, CG : CE = CD : CP, and therefore, becauſe CE is given, the point G is a given one, that is, D is a given point.

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Hence for the ſum of the ſquares.

Join the given points A, B, and biſect AB at E, and take the line EP ſuch, that $\frac{1}{2}AB^2 + 2PE^2 =$ the given ſpace; join CE and make EG to EC in the given ratio, and to the given circle, apply GD a 4th proportional to CE, CG, and EP; then join CD and it is done.

For the difference of the ſquares.

Demit upon AB the $\perp$ PF, join CF, and let DH, drawn parallel to PF, meet CF at H. Then ſince $AP^2 - PB^2 = AF^2 - FB^2 =$ a given ſpace, the point F will be given, ſince AB is a given line. Alſo, becauſe FH : FC = PD : PC, is a given ratio, the point H will be given. Whence in this caſe, join AB and divide it in F, ſo that $AF^2 - FB^2$ may be $=$ to the given ſpace; join CF, and take FH : FC = the given ratio, and draw HI $\perp$ to AB, meeting the circle in D: then joining C, D, it is done.

Meſſrs. Cunliffe *and* Whitley *ſent neat ſolutions to this queſtion.*

X. QUESTION 250, *anſwered by Mr.* Cunliffe.

The area and the perimeter of the polygon being given the radius of the inſcribed circle becomes known; for the area is equal to a rectangle under half the perimeter and the radius of the inſcribed circle. Therefore let r denote the radius of the inſcribed circle, $2t$ one of the ſides of the polygon, n the number of its ſides, and p the given perimeter thereof. Let alſo, z denote the length of that arc of the inſcribed circle whoſe tangent is t, and put $c = 3{\cdot}14159$ &c.

Then it is obvious that $2nt = p$, and $2nz = (p \div t)\, z = 2rc$, the circumference of the inſcribed circle. Moreover it is well known that

$$z = t - \frac{t^3}{3r^2} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \&c.;\ \text{wherefore,}$$

$$2nz = (p \div t)\, z = (p \div t) \times \left(t - \frac{t^3}{3r^2} + \frac{t^5}{5r^4} - \frac{t^7}{7r^6} + \&c.\right)$$

$$= p \times \left(1 - \frac{t^2}{3r^2} + \frac{t^4}{5r^4} - \frac{t^6}{7r^6} + \&c.\right)$$

$$= 2rc;\ \text{or}$$

$$1 - \frac{t^2}{3r^2} + \frac{t^4}{5r^4} - \frac{t^6}{7r^6} + \&c. = 2rc \div p;\ \text{whence,}$$

$$t^2 - \frac{3t^4}{5r^2} + \frac{3t^6}{7r^4} - \frac{3t^8}{9r^6}\ \&c. = 3r^2 - \frac{6r^2c}{p} = q,\ \text{a given quan-}$$

tity,

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tity, because both r and p are given; and by reverting the series

$$t = q^{\frac{1}{2}} + \frac{3q^{\frac{3}{2}}}{10r^2} + \frac{141q^{\frac{5}{2}}}{1400r^4} + \frac{1411q^{\frac{7}{2}}}{42000r^6} + \&c.$$

$$= q^{\frac{1}{2}} \times (1 + \frac{3q}{10r^2} + \frac{141q^2}{1400r^4} + \frac{1411q^3}{42000r^6} + \&c.)$$

Now t being determined, n will be found from the equation $n = p \div 2t$.—In all cases when p, the perimeter of the polygon does not much exceed $2rc$, the circumference of the inscribed circle, the preceding series expressing the value of t will converge fast, because in those cases the value of q will be but small. To prove this, put $2rc + v = p$, then

$$q = 3r^2 - \frac{6r^3c}{p} = 3r^2 - \frac{6r^3c}{2rc + v} = \frac{3r^2v}{2rc + v},$$ which, because v is small, will evidently be small.

But the solution of any particular case of the question may likewise be easily made out from what is done above by the method of trial and errour. For it has been found that $pz \div t = 2rc$, or $z \div t = 2rc \div p$, that is, the ratio of an arc of a circle to its tangent is given, from whence, by a few trials, the arc itself, as well as the tangent, may be found, and from thence n as before.

XI. QUESTION 251, *answered by Mr.* Lowry.

Plate 30, Fig. 582. Let CB be the given distance of the illuminated point from the point directly under the candle, and BA the required height of the candle. Join CA, and from B upon CA drop the $\perp$ BD.

Now the effect of a particle of light to illuminate an object being directly as the sine of the angle of incidence, and reciprocally as the square of the distance, or directly as BD, and inversely as CA^2 we have $BD \div AC^2$ a maximum. But by sim. $\triangle$s $CA^2 : CB^2 :: CB^2 : CD^2$; therefore $BD \div AC^2 = (CD^2 \times BD) \div CB^4$; but CB^4 is a constant quantity, therefore when $BD \div AC^2$ is a maximum, $CD^2 \times BD$ will be so too; and by Simpson on the max. et min. this will happen when $CD^2 = 2BD^2$, or when $CB^2 = 2BA^2$; therefore $BA : BC :: 1 : \sqrt{2}$, or as the side of a square to its diagonal.

The same, answered by Tyro Philomatheticus.

Let $a =$ the given distance of the point to be illuminated from the point directly under the candle, and $x =$ the required altitude.

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Then it is well known that $\frac{1}{a^2 + x^2}$ will be as the number of particles, and the force of each being as the fine of the angle of incidence, we have $\frac{x}{\sqrt{(a^2+x^2)}}$ as the force of one particle; therefore $\frac{x}{\sqrt{(a^2+x^2)}} \times \frac{1}{a^2+x^2} = \frac{x}{(a^2+x^2)^{\frac{3}{2}}}$, is as the force or power of the flame to illuminate the faid point, which by the queftion, muft be a *maximum*. Put into fluxions and reduced gives $x = a\sqrt{\frac{1}{2}}$.

XII. QUESTION 252, *anfwered by* Tyro Philomatheticus.

From a little conſideration, it will manifeſtly appear that the length of the curve will be found by the expreſſion given for the involute of a circle; that is, putting $p = 3{\cdot}141592$ &c.

$$\left\{ f - (\tfrac{a}{2p} + \tfrac{1}{2}b)^2 \right\} \div (\tfrac{a}{p} + b) = \text{the length of the curve.}$$

XIII. QUESTION 253, *anfwered by Mr.* Lowry.

By the ſcholium to Prop. XXXV, Emerſon's Mechanics. "A body projected on an inclined plane will deſcribe a parabola. And if the velocity of projection upon the plane to be the velocity of a projectile in the air; as the relative gravity on the plane to the abſolute gravity; and both projected at the ſame obliquity, the ſame parabola will be deſcribed in both caſes." Now the relative gravity on the plane is to the abſolute gravity, as the ſine of the angle of elevation to radius, that is, as ſine of 10° : radius :: 8 feet (the velocity of projection on the plane) to 46·0701 feet, the velocity neceſſary to deſcribe the ſame parabola in free ſpace; and $(46{\cdot}0701)^2 \div (4 \times 16\frac{1}{12}) = 32{\cdot}99151$ feet, the ſpace which the ball muſt deſcend to acquire that velocity. Suppoſe the plane of the table to be placed in a vertical direction, A the point from whence the ball is projected, (Plate 30, Fig. 587) K the centre of the table, and F the place where the ball quits it. Draw KIS, QABD, $\perp$ to the horizon, and take $AQ = QQ' = 32{\cdot}99151$; draw the indefinite right lines QSC, Q'C'C, and through A draw the tangent AC', to touch the circle at A and meet Q'C' at C', and from C' to SK, apply C'u = C'A; then draw Au, and it is the direction required, as is evident from Prop. IV. Page 37 and 38 of Burrow's Elegant

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Elegant Little Eſſay on Projectiles; and by the rules there given the angle KAV is found $= 30° 41' 35''$. Again with the diſtances KS, AQ, and centres K, A, deſcribe two arcs interſecting in P: then P will, by the ſame eſſay, be the focus of the parabola, and CQO its directrix.

Join AG, AK, KP, and KF, and draw OPN parallel, and AIL, KBM, and FEDN perpendicular to ADQ. Then by the property of the circle $GI = AG^2 \div 2GK = 416'$ and $IR = 239{\cdot}583'$, $AI = 9{\cdot}992$, and by trigonometry $\angle AKI = 4° 46' .25''$. And in the $\triangle APK$ the three ſides are given to find $\angle AKP = 3° 59'$; therefore the $\angle PKM$ is found $= 81° 14' 35''$, and KM and PM $= 78{\cdot}488$ and $509{\cdot}39$ reſpectively. Hence KP $-$ PM $=$ PO $= 6{\cdot}091 =$ half the latus rectum $= \frac{1}{2}p$.

Put $a =$ KM or HE, $b =$ KF, and $x =$ FE.

Then by the property of the parabola

$p : 2KM + FE :: FE : EK$, or

$p^2 : (2a + x)^2 :: x^2 : b^2 - x^2$; therefore

$p^2b^2 - p^2x^2 = 4a^2x^2 + 2ax^3 + x^4$, or

$x^4 + 2ax^3 + (4a^2 + p^2)\,x^2 = p^2b^2$, and from the reſolution of this equation x is found $= 9{\cdot}027$, and MN $= 119{\cdot}66$. And by Dr. Hutton's Menſuration the arc HF is found $= 735{\cdot}083$, the arc AH $= 458{\cdot}767$, and their difference AF $= 276{\cdot}316$ the length of the track required.

XIV. QUESTION 254, *anſwered by* Collator, *the Propoſer.*

Plate 30, Fig. 583. Let ABCD be the ellipſe, F and f its foci, AC and BD its major and minor axes. Take any point E between A and B, join Ef, EF. Now, ſince heat is inverſely as the ſquare of the diſtance, the heat coming from F will be as $\frac{1}{(FE)^2}$, and the heat coming from f will be as $\frac{1}{(fE)^2}$; therefore the whole heat will be as their ſum, or as $\frac{1}{(FE)^2} + \frac{1}{(fE)^2}$; but whilſt FE is greater than fE the ſquare of FE decreaſes faſter than that of fE increaſes, becauſe the ſum of FE, and fE is always the ſame, and therefore FE decreaſes juſt as faſt as fE increaſes; conſequently $\frac{1}{(FE)^2}$ increaſes faſter than $\frac{1}{(fE)^2}$ decreaſes. Hence, whilſt

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whilſt E is between A and B the ſum $\frac{1}{(FE)^2} + \frac{1}{(fE)^2}$ increaſes as the point E approaches to B, and decreaſes as the ſame point recedes from B; conſequently the heat will be a maximum at B, and a minimum at A.

The ſame, anſwered by Mr. Lowry.

Let AEBCD be the given ellipſe, F and f the foci, and F any point in the curve. Join fE, FE. Then, ſince the effect of heat is reciprocally as the ſquare of the diſtance, the heat at F will be as $\frac{1}{(FE)^2}$, and that at f as $\frac{1}{(fE)^2}$, and therefore the whole effect will be as $\frac{1}{(FE)^2} + \frac{1}{(fE)^2}$; and ſince CF + CG is given (= the tranſverſe diameter AC) it is evident $\frac{1}{(FE)^2} + \frac{1}{(fE)^2}$ will be a maximum when $FE = fE$, and a minimum when fE exceeds FE by the greateſt diſtance poſſible, or when fE coincides with AC; therefore the greateſt heat will be at the extremity of the conjugate diameter, and the leaſt heat at the extremity of the tranſverſe diameter.

Solutions to this queſtion were alſo received from Meſſrs. Francis *and* Tyro Philomatheticus.

XV. QUESTION 255, *anſwered by Mr.* Cunliffe.

Plate 30, Fig. 584. Let ACD be drawn through the centres C and O, of both circles; join DG and produce it till GM = GD. Draw ME to touch the circle whoſe centre is C in E; through C and E draw the chord CEP, upon which demit the ⊥ GV, and EP will be biſected in V.

For, draw MQ ∥ to EP, and meeting DP in Q, and produce VG to meet MQ in N. Then DP is at right angles to CP, therefore GN and DQ are parallel: but GM = DG by the conſtruction, therefore MN = NQ, or their equals, EV = VP. *Q. E. D.*

If it had been required to draw the chord CP, ſuch that demitting the ⊥ GV; EV and VP might have a given ratio, the conſtruction would not have been materially different.

For

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For in that case it had only been taking GM to DG in the given ratio, and the remaining part of the construction the same as before.

The same, answered by Mr. Johnson.

CONSTRUCTION. Plate 30, Fig. 585. On the line joining the points O, G, describe a semi-circle, and apply therein OK equal to half the radius of the circle whose centre is C; draw CP ‖ to OK and it is done.

DEMONSTRATION. Draw OI ⊥ to CP, and produce GK to meet CP at L; then we have to prove that EL = LP. Now by the property of the circle CI = IP, or CE + EI = IL + LP, and by construction 2OK or 2IL = CE; therefore 2IL + EI = IL + LP, and taking IL from each, there remains EL = LP. *Q. E. D.*

If 2OG be less than CE the problem is impossible.

The same, answered by Mr. Swale, *the Proposer.*

CONSTRUCTION. Plate 30, Fig. 586. Join CG, OG, and in OG take the point L such that $GL^2 - LO^2$ = the difference between the square of CG and the sum of the squares of the radii CO, CA; draw LE ⊥ to GO meeting the former circle in E; and through C and E draw CP and it is done.

DEMONSTRATION. By the construction,

$GL^2 - LO^2 = CG^2 - CO^2 - CA^2 = GE^2 - EO^2$, that is,

$GE^2 + EC^2 - EO^2 = CG^2 - CO^2 = CG^2 - OI^2$.

But, because GF is perpendicular to CP, we have,

$CG^2 = GE^2 + EC^2 + CE \times 2EF$; or

$GE^2 + EC^2 - EO^2 = GE^2 + EC^2 + CE \times 2EF - OI^2$:

Then, $OI^2 - EO^2 = CE \times 2EF$.

Now, since HO = OI, it follows that,

$OI^2 - OE^2 = EI \times EH$; Hence

$CE \times 2EF = EI \times EH = CE \times EP$, by the circle;

Then 2EF = EP, or EF = FP;

that is EP is bisected at F by the ⊥ GF. *Q. E. D.*

Ingenious solutions were also given by Messrs. Lowry *and* Whitley.

XVI. QUESTION 256, *answered by Mr.* John Whitley.

Plate 30, Fig. 588. Let lines be drawn as directed by the question, and join OB, OC, GC. By known properties

AC^2

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Draw AF parallel to DB, and CG parallel to EB, each terminating in BB'; alſo upon ED let fall the perpendicular BH.

By the queſtion, $AB = A'F = a$; $BC = C'G = c$; $AA' = BF = p$; $CC' = BG = r$; whence $B'F = q - p$ and $B'G = q - r$; and becauſe of the parallels A'F, DB.

$B'F : A'F :: B'B : D'B = (A'F \cdot B'B) \div B'F = aq \div (q - p)$.

Alſo becauſe of the parallels C'G, EB,

$B'G : C'G :: B'B : EB = (C'G \cdot B'B) \div B'G = cq \div (q - r)$.

Now in the $\triangle$ ABC all the ſides are given; whence the $\angle$ ABC may be found by trigonometry. Moreover in the $\triangle$ DBE, the ſides DB and EB, together with their included $\angle$ DBE, are given; to find the $\angle$ EDB or its ſupplement, the $\angle$ HDB; and from thence the complement of this, or the $\angle$ DBH becomes known.

Suppoſe BM to be a meridian paſſing through B: the $\angle$ ABM is given by the queſtion; therefore, as the $\angle$ DBH is given, the $\angle$ HBM becomes known, or the angle becomes known which the direction of the dip of the ſtratum makes with the meridian. Now HB may be eaſily determined from what has already been ſhewn: and at the length HB the ſtratum manifeſtly dips the depth BB'.

XX. QUESTION 260, *anſwered by* Collator, *the Propoſer.*

Let A be the weight of the ſyſtem, a the radius of the cylinder, x the diſtance of the centre of gyration from the centre of gravity, $m = 16\frac{1}{12}$, V the velocity generated in the time T, and S the ſpace paſſed over by the weight P.

Now, by the property of the centre of gyration, if all the matter in the ſyſtem were collected, and uniformly diſpoſed over the circumference of a circle whoſe radius is x, and whoſe plane is perpendicular to the axis of motion, and paſſing through the centre of gravity of the ſyſtem; the ſame force would generate the ſame angular velocity in the ſame time: alſo if inſtead of the matter being diſpoſed at the diſtance of the centre of gyration, an equivalent maſs be ſubſtituted at the ſurface of the cylinder, the ſame angular velocity will ſtill be preſerved; which maſs will be expreſſed by $\frac{x^2 A}{a^2}$.

By Atwood on *Rectilinear Motion and Rotation*, p. 36, $\frac{Q}{q} = \frac{M}{m} \times \frac{v}{V}$; and, in this caſe, $M = m = P$; therefore $\frac{Q}{q} = \frac{v}{V}$, and

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and $Q = N = \frac{Ax^2}{a^2}$; and the whole quantity of matter, moreover,

$= \frac{Ax^2}{a^2} + P = \frac{Ax^2 + Pa^2}{a^2} = Q$; but $\frac{Q}{q} = \frac{v}{V} = \frac{2m}{V}$, and

$q = P$, therefore $\frac{Q}{q} = \frac{Q}{P} = \frac{Ax^2 + Pa^2}{Pa^2} = \frac{2m}{V}$.

Hence $V = \frac{2m\,Pa^2}{Ax^2 + Pa^2} = 2\sqrt{(mS)}$. Therefore

$2\sqrt{(mS)} \times Ax^2 + 2\sqrt{(mS)} \times Pa^2 = 2m\,Pa^2$, or
$2\sqrt{(mS)} \times Ax^2 = 2mPa^2 - 2\sqrt{(mS)} \times Pa^2$, and

$$x^2 = \frac{2mPa^2 - 2\sqrt{(mS)} \times Pa^2}{2\sqrt{(mS)} \times A} = \frac{2Pa^2 \times \left\{ m - \sqrt{(mS)} \right\}}{2\sqrt{(mS)} \times A}.$$

Consequently $x = \pm \sqrt{\frac{2Pa^2 \times \left\{ m - \sqrt{(mS)} \right\}}{2\sqrt{(mS)} \times A}}$, the distance of gyration from the centre of gravity; the latter of which being determinable by well known methods, it is unneceſſary to purſue the inveſtigation further.

XXI. QUESTION 261, *anſwered by Mr.* Cunliffe, *the Propoſer.*

Fig. 591, Pl. 30. Let DBA repreſent the catenarian curve AD its axis, and DB its greateſt ordinate, about which it revolves in the generation of the ſolid.

Draw any ordinate QP, and alſo the line PR parallel to AD meeting DB in R. Put l = the length of the curve APB, AD $= b$, AQ $= x$, QP $= y$, a = the tenſion of the curve at A, and $p = 3{\cdot}1415926$, Then by the property of the curve

$\dot{y} = \frac{a\dot{x}}{\sqrt{(2ax + x^2)}}$;—and by the writers on fluxions

$$(RP)^2 \times p \times \text{flux.DR} = \frac{p \times (b-x)^2 \times a\dot{x}}{\sqrt{(2ax + x^2)}}$$

$$= ap \times \left(\frac{b^2\dot{x}}{\sqrt{(2ax+x^2)}} - \frac{2bx\dot{x}}{\sqrt{(2ax+x^2)}} + \frac{x^2\dot{x}}{\sqrt{(2ax+x^2)}} \right)$$

will expreſs the fluxion of the ſolid generated by the revolution of the ſpace DAPR about the ordinate DB: the fluent of which is

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$$ap \times \left(\frac{x}{2}\sqrt{(2ax+x^2)} - (\frac{3a}{2}+2b)\sqrt{(2ax+x^2)}\right)$$

$$+ ap \times (b^2 + \frac{3a^2}{2} + 2ab) \times \text{h. l.} \left(\frac{a+x+\sqrt{(2ax+x^2)}}{a}\right);$$

and this expreſſion is correct, for the whole will vaniſh or become $= o$, when $x = o$, as it ought.

But when $x = b$, the expreſſion becomes

$$ap \times (b^2 + \frac{3a}{2} + 2ab) \times \text{hyp. log.} \left(\frac{a+b+\sqrt{(2ab+b^2)}}{a}\right)$$

$$- ap \times (\frac{3a+3b}{2})\sqrt{(2ab+b^2)}$$ for the whole content of the the ſolid generated by the revolution of the ſpace DPBA about the ordinate DB.

Again, by the property of the curve $a = (l^2 - b^2) \div 2b$; let this be written inſtead of a in the preceding expreſſion for the ſolidity and it becomes

$$p \times (\frac{l^2-b^2}{2b}) \times (\frac{3(l^2-b^2)^2}{8b^2} + l^2) \times \text{h. l.} (\frac{l+b}{l-b}) - \frac{3lp}{8} \times (\frac{l^4-b^4}{b^2})$$

which is to be a *maximum* by the queſtion, or

$$\left\{\frac{3}{2}(\frac{l^2-b^2}{2b})^2 + l^2 (\frac{l^2-b^2}{2b})\right\} \times \text{h. l.} (\frac{l+b}{l-b}) - \frac{3l}{2} (\frac{l^2-b^2}{4b^2})$$

muſt be a maximum: therefore conſidering b variable and putting the fluxion of the expreſſion $= o$, there will be had

$$\left.\begin{array}{l} -\dot{b} (\frac{l^2+b^2}{2b^2}) \times \left\{\frac{3}{2} \times (\frac{l^2-b^2}{2b})^2 + l^2\right\} \times \text{h. l.} (\frac{l+b}{l-b}) \\ + \dot{l}b \times (\frac{3(l^2-b^2)^2}{8b^3} + \frac{l^2}{b}) + 3l\dot{b} \times (\frac{l^4+b^4}{4b^3}) \end{array}\right\} = o;$$

and dividing by $\frac{b}{b^2}$

$$\left.\begin{array}{l} -(\frac{l^2+b^2}{2}) \times \left\{\frac{3}{2} \times \left(\frac{l^2-b^2}{2b}\right)^2 + l^2\right\} \times \text{h. l.} (\frac{l+b}{l-b}) \\ + \dot{l} \times \left\{\frac{3(l^2-b^2)^2}{8b} + l^2 b\right\} + 3l (\frac{l^4+b^4}{4b}) \end{array}\right\} = o; \text{ or}$$

$$\left.\begin{array}{l} -(\frac{l^2+b^2}{2}) \times (\frac{9l^4 - 10l^2b^2 + 9b^4}{8b^2} \times \text{h. l.} (\frac{l+b}{l-b}) \\ + \dot{l}b \times (\frac{9l^4 + 2l^2b^2 + 9b^4}{8b^2}) \end{array}\right\} = o.$$

From

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From whence h. l. $(\frac{l+b}{l-b}) = \frac{2lb \times (9l^4+2l^2b^2+9b^4)}{(l^2+b^2) \times (9l^4-10l^2b^2+9b^4)}$,

which expreſſion is well adapted for finding the value of b by the method of trial and error; and by a few approximations of this ſort we get $b = l \times .7864616$, which is correct to the laſt decimal place

Again, from the property of the curve,

$DB = (\frac{l^2-b^2}{2b}) \times$ h. l. $(\frac{l+b}{l-b}) = l \times .5151723333$, or

$CB = l \times 1.030344666$. *W. W. R.*

☞ All the properties of the catenary mentioned in the ſolution to this and queſtion 265 may be ſeen, together with their inveſtigations, in Dr. Hutton's Mathematical Dictionary.

XXIII. QUESTION 263, *anſwered by* Collator, *the Propoſer.*

Plate 31, Fig, 603. Let ADB be a great circle of the earth or the equator; C its centre; AF the ſpiral. Put $r = AC =$ rad.; $m = 16\frac{1}{12}$ feet; and the force of gravity at the ſurface of the earth $= 1$.

The velocity acquired by a body in falling through $\frac{1}{2}r$, by the force of gravity, will be $=$ the velocity with which a body may deſcribe a circle at the diſtance AC; therefore

$\sqrt{m} : \sqrt{\frac{r}{2}} :: 2m : \frac{2m}{\sqrt{m}} \times \sqrt{\frac{r}{2}} = \sqrt{(2mr)} =$ the velocity in the circle ADB per ſecond.

Let n be the number of feet in the circumference of the earth, then $\sqrt{(2mr)} : n :: 1 : n \div \sqrt{(2mr)} =$ period. time in the circle ADB, which put $= p$. Let P be the period. time of the earth's revolution, and ſince C, the centrifugal force, is as $\frac{r}{p^2}$, and r is here given, C is as $\frac{1}{p^2}$, $P^2 : p^2 :: 1 : p^2 \div P^2 = a$, the centrifugal force at A. Again (if $x =$ diſt. CF) ſince C is as $\frac{x}{p^2}$, and p is given, C is as x, and therefore $r : x :: 1 : \frac{x}{r} =$ the centripeta

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force at F; alſo $r : x :: a : \frac{ax}{r} =$ the centrifugal force at F, whence the effective force tending to the centre C $= \frac{x}{r} - \frac{ax}{r}$ $= \frac{x}{r} \times (1-a) = x \times \frac{1-a}{r} =$ F. Now, from the general equation, $Fx = v\dot{v}$, we may derive a value for v in terms of x, for by ſubſtitution, $x\dot{x} \times \frac{1-a}{r} = v\dot{v}$, and $x^2 \times \frac{1-a}{r} = v^2$; and therefore $v = x\sqrt{\frac{1-a}{r}}$: whence v is as x, and therefore the ſpiral is the logarithmic one.

XXIV. QUESTION 264, *anſwered by Mr.* Lowry.

Pl. 30, Fig. 602. Let A, B be the given points, EF the given chord, and O the centre of the given circle. About the centre O deſcribe a circle to touch EF at G, then OG being drawn will be ⊥ to EF, and of a given length; ſuppoſe the ⊥s AC, BD to be drawn as required. Then,

Firſt. *When the ratio of BD to AC is given.*

Let EF and BA be produced till they meet at H.

Becauſe AC : BD :: AH : AB, it follows, by diviſion, that the ratio of AH : AB is given, and AB is given in magnitude, therefore AH is given, and H is a given point. Conſequently if HEF be drawn to touch the circle at H, the thing is done.

Secondly. *When the ſum or difference of AC and BD are given.*

Theſe caſes are conſtructed generally at Article 28, Vol. I. of the Repoſitory, and a reference to that article is all that is neceſſary in this place.

Thirdly. *When the rectangle BD · AC is given.*

Through the centre O draw IOS parallel to EF, meeting AC in I, and BD in S; join AO and draw OK ⊥ to AB, KT ⊥ to OI, KL ⊥ to AO, and AV ⊥ to KT. Then
BD = BS + OG and AC = AI + OG, therefore
BD·AC = BS·AI + OG·(BS + AI) + OG^2, and therefore
BS·AI + OG · (BS + AI) + is equal to a given ſpace, equal, ſuppoſe, to AO·AZ. Now
TK is evidently equal to half the ſum of BS, AI, and

KV

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KV equal to half their difference; therefore
BS = TK + KV, and AI = TK — KV; wherefore
BS·AI = TK² — KV² and TK² — KV² + 2OG·TK = AO·AZ, or
LO·TK² — LO·KV² + 2OG·LO·TK = AO·LO·AZ:

But the triangles AKV, KOT are similar, therefore
AL : LO :: AK² : KT² :: KV² : TO², or
LO·KV² = AL·TO² = AL·(OK² — KT²)
= AL·LO·OL — AL·KT², wherefore
LO·TK² + AL·TK² + 2OG·LO·TK } = AO·LO·AZ + AL·AO·OL, or
AO·TK² + 2OG·LO·TK = AO·LO·LZ,
Make AO·OQ = 2OG·LO, then
AO² + TK² + AO·OQ·TK = AO·LO·LZ, or
TK² + OQ·TK = LO·LZ, therefore TK becomes known by the 29th. VI. of the Elements, (See Playfair's Edition,) and the method of conſtruction is obvious.

Fourthly. *When the ſum of the ſquares of BD, AC is given.*
BD² + AC² = BS² + AI² + 4OG·KT + 2OG²,
therefore BS² + AI² + 4OG·KT is given:
but BS = TK + VK, and AI = TK — KV,
therefore BS² + AI² = 2TK² + 2KV², and conſequently
TK² + KV² + 2OG·KT is given, = (ſuppoſe,) to AO·AM,
that is, LO·TK² + LO·KV² + 2OG·LO·KT = AO·AM·LO.
But by ſimilar triangles,
AL : LO :: AK² : KO² :: KV² : TO² = OK² — TK²;
therefore LO·VK² = AL·(OK² — TK²), wherefore
LO·TK² + AL·(OK² — TK²) } = AO·AM·LO, or
+ 2OG·LO·KT }
(LO — AL)·TK² + 2OG·LO·KT = AO·AM·LO — AL·KO²
= AO·LO·LM.
Make 2OG·LO = (LO — AL)·OX,
and LO·LM = (LO — AL)·OY.
Hence TK² + OX·TK = AO·OY.

Conſequently TK may be determined as before.

Finally. *When the difference of the ſquares of BD, AC is given.*
BD² — AC² = BS² — AI² + 2OG·(BS — AI)
= 4TK·VK + 4OG·VK;
But the ratio of OT to VK is given;
therefore (TK + OG)·OT is given;
Draw RW ⊥ to KO produced at W; then
OK : OT :: KR·(TK + OG) : RW, or
OK·RW = (TK + OG)·OT; therefore RW is a given line, and the *locus* of the point R is a ſtraight line RU drawn parallel to KO. Alſo the locus of the point T is a ſemi-circle deſcribed on OK; it is therefore required to draw KR ſo that the inter-

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cepted part TR may be of a given length, which is a solid problem, and may be effected by the Conic Sections, or (much easier) by the conchoidial ruler.

XXV. QUESTION 265, *answered by Mr.* Cunliffe, *the Proposer.*

Fig. 592, Pl. 30. Let APB represent a position of the chain, and imagine SV to be drawn ⊥ to the middle of AB, meeting the curve of the chain in V: also let PQ and PR be drawn ⊥ to SV and AB respectively.

Put $2l$ = APB the given length of the chain, or l = APV, m = the length of the part PV, SV = b, AR = x, RP = SQ = y; also let a denote the tension of the chain at V when in the position AVB.

Then VQ = $b - y$, and by the property of the curve,

$$a = \frac{m^2-(b-y)^2}{2(b-y)} = \frac{l^2-b^2}{2b}, \text{ whence } bm^2 + b^2y - by^2 = bl^2 - l^2y,$$

which being properly ordered gives

$$b = \frac{l^2+y^2-m^2}{2y} - \frac{\sqrt{\left\{(l^2-y^2)-2m^2(l^2+y^2)+m^4\right\}}}{2y}.$$

Again, by the property of the curve,

$$AS = a \times \text{h. l.}\left(\frac{AV+SV}{AV-SV}\right) = a \times \text{h. l.}\left(\frac{l+b}{l-b}\right), \text{ and}$$

$$PQ = RS = a \times \text{h. l.}\left(\frac{PV+QV}{PV-QV}\right) = a \times \text{h. l.}\left(\frac{m+b-y}{m+y-b}\right); \text{ whence}$$

$$AS - RS = AR = x = a \times \left\{\text{h. l.}\left(\frac{l+b}{l-b}\right) - \text{h. l.}\left(\frac{m+b-y}{m+y-b}\right)\right\}$$

$$= a \times \text{h. l.}\frac{(l+b)\times(m+y-b)}{(l-b)\times(m+b-y)}$$

$$= \frac{l^2-b^2}{2b} \times \text{h. l.}\frac{(l+b)\times(m+y-b)}{(l-b)\times(m+b-y)},$$

by substituting for a its equal $\frac{l^2-b^2}{2b}$.

And exterminating b from this equation by means of its value found above, there will be had an equation expressing the nature of the

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the curve, which is the locus of the point P, in terms of its own abſciſſa and ordinate.

The area of the curve may be generally expreſſed by ſeries.

XXVI. QUESTION 266, *anſwered by Mr.* Lowry.

Plate 30, Fig. 593, 594. ANALYSIS. Suppoſe it done, and let A and O be the centres of the given circles, B the given point in the line joining their centres, D the point of contact, and C the centre of the required circle. Draw CA and CB; then CA will paſs through D, the point of contact, and CD will be = CB. Therefore AC + CB = AD in fig. 593, and AC — CB = AD in fig. 594, = to a given line.

Draw the radius CO, then (Stewart's General Theorems, Prop. II.) AC^2 together with the ſpace to which BC^2 has the ſame ratio that BO has to AO, is equal to the rectangle BAO together with the ſpace to which CO^2 has the ſame ratio that BO has to AB, that is equal to a given ſpace, equal to P^2.

Therefore the problem is reduced to the following, viz. Given the ſum or difference of two lines to determine them, when the ſquare of one, together with the ſpace to which the ſquare of the other has a given ratio, is equal to a given ſpace; which may be effected as follows:

Take ad = the given ſum or difference (ſee fig. 595 and 596), and on ad (produced when the difference is given) take any point b and erect the perpendicular bg ſo that $db^2 : bg^2$ in the given ratio of BO : AO; draw dg and from a apply am = to P; draw alſo $mc \perp$ to ad; ſo ſhall ac, cd be the two lines required; that is, if AC be made = ac, the point C will be determined.

For cm^2 is evidently the ſpace to which cd^2 or CB^2 has the ſame ratio that BO has to AO, and $ac^2 + cm^2 = AC^2 + cm^2 = P^2$ = the given ſpace.

XXVII. QUESTION 267, *anſwered by Mr.* Lowry, *the Propoſer.*

Plate 30, Fig 597. CONSTRUCTION. Biſect the given baſe AB at D, and draw DO $\perp$ to AB, and ſuch that $\frac{1}{4}$DO may be to DB in the given ratio; on DO deſcribe a ſemicircle, and therein apply DC = DB; join AC, BC, and ACB is the $\triangle$ required.

About the centre, with the diſtance OCO, deſcribe a circle cutting DO produced in E and F. Then we ſhall prove,

1ſt. That the circle ECF is the locus of the vertices of all the

$\triangle$s

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△s conſtituted on the baſe AB when the ſum of the ſquares of the ſides has to the area of the △ the ratio of 4DO to DB.

2ndly. That C is the point in the circumference ECF where the ratio of AC to BC is the greateſt poſſible.

Let C′ be any point in the circumference; join AC′, BC′, OC′, and DC′, and draw CH parallel, and C′G perpendicular to AB. Then

$OD^2 - DB^2 = OC^2$ or $C'O^2 = C'H^2 + HO^2$, or

$OD^2 - OH^2 (= 2OH \cdot HD + HD^2) = C'H^2 + DB^2$,

and adding HD^2 to each,

$2OD \cdot HD = C'H^2 + DB^2 + HD^2 = DB^2 + C'D^2$; therefore

$4OD \cdot HD = 2DB^2 + 2C'D^2 = C'A^2 + C'B^2$. Conſequently

$C'A^2 + C'B^2 (4OD \cdot HD)$: area of the △AC′B(BD·DH)::4OD:DB.

And if the points C′ and C coincide

$AC^2 + BC^2$: area △ ACB :: 4OD : DB, that is, in the given ratio.

Again, produce OC to meet AB produced in S, then ſince AD = DB = DC, and the ∠DCS a right one, CS is a tangent to the circle which paſſes through the points A, C, B. With the centre S, and diſtance SC deſcribe a circle, cutting BC′, in c, and join Ac.

Then $SC^2 = BS \cdot SA$, and by a well known propoſition

AC : CB :: Ac : Bc; but

the ratio of Ac : Bc is evidently greater than that of AC : BC; therefore the ratio of AC : CB is greater than that of AC′ : BC′.

The triangle ACB is right angled at C.

XXVIII. QUESTION 268, *anſwered by Mr.* Lowry.

Plate 30, Fig. 598. CONSTRUCTION. On the given line AD deſcribe a ſegment of a circle to contain an angle equal to half the given vertical angle. At A draw a tangent AI of ſuch a length that its ſquare may be equal to the given difference. Through the point I deſcribe a circle concentric with the other, and from B, the extremity of the diameter drawn perpendicular to AD, draw BC to make the ∠ BCA = the given one; ſhall ACB be the △ required.

Take CF = BC, and join BF, DE, and from B draw the tangent BH. Then ſince CF = BC, AF is equal to the ſum of the ſides of the △ ACB, and the ∠ CFB = CBF = half the vertical ∠ ACB, =, by conſtruction, to the ∠ AED, and the ∠ EAD is common to the △s ABF, ADE; therefore they are ſimilar, and

$AD \times AF = AE \times AB$: but,

$AI^2 = BH^2 = BE \times AB$; therefore

$AD \times AF + AI^2 = AE \times AB + BE \times AB = AB^2$.

Therefore

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Therefore the ſquare of the baſe exceeds the rectangle under the ſum of the ſides and and a given line, by the given difference.

The perpendicular BK is evidently the greateſt that can be inſcribed in the circle when it paſſes through the centre, and therefore in this caſe muſt be the greateſt poſſible.

XXIX. QUESTION 269, *anſwered by Mr.* Lowry, *the Proposer.*

This problem admits of three caſes according to the ſituation of the given points with reſpect to circle; viz.

1ſt. When the given points are without the circle.

2ndly. When two of them are within, and the remaining one without the circle; and

3rdly. When one of the points is within and the other two without the circle.

The conſtruction for the firſt caſe is as follows. Fig. 599, Pl. 30.

Let A, B, C, be the three given points, KHLM the circle in which the trapezium is to be inſcribed, and IS the other given circle. Draw the tangents AD, BE, and in AB aſſume the point F (on the ſame ſide of A with the point C) ſo that the rectangle $CAF = AD^2$, and in AB find the point G (on the ſame ſide of B with the point F) ſo that the rectangle $GBF = BE^2$; draw GM to touch the given circle IS at I, alſo draw CM meeting the circle at K, AK meeting the circle at H, BH meeting the circle at L, join LM, and HKLM is the trapezium required.

Again, for the 2nd and 3rd Caſes. Fig. 600, 601, Pl. 30.

In AB aſſume the point F ſo that the rectangle $CAF = AD^2$, and the point G ſo that the rectangle $GBF = PBQ$; draw GM to touch the given circle IS at I, alſo draw CM meeting the circle at K, AK meeting the circle at H, BH meeting the circle at L, join LM, and HKLM is the trapezium required.

For, by Prop. XLV. and XLVI. Lib. I, of Dr. *Stewart's Propoſitiones Geometricæ*, the points L, M, G are in a ſtraight line, therefore the ſides HK, KL, and KM paſs through the given points A, B, C, reſpectively, and the ſide ML touches the given circle.

Q. E. D.

XXX.

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XXX. Or, PRIZE QUESTION 270, *answered by the Propoſer*, Philalethes Cantabrigienſis.

In printing this queſtion, the ſine + was put inſtead of the ſine — before the term $2\sqrt[3]{x}$, (as was noticed on the cover of No. XII.) which will occaſion a great difference in the value of y, and incumber the numerical calculation with decimals, but the method of ſolution will be nearly the ſame. I ſhall therefore give a ſolution to the queſtion as it was written.

By *Madam* AGNESI's *Analytical Inſtitutions*,* Vol II, Book III, Sect. I, Article 55, put $u^6 = x$; then we ſhall have $\frac{6iu^5}{(1-2u^6+u^{12})^2} = y$; and, ſince the denominator of this fraction is the product of $(1+u-u^6)^2 \times (1-u)^6$, we ſhall have by Art. 62 of the ſame Section of the book before mentioned,

$$\frac{6iu^5}{(1-2u^6+u^{12})^2} = \left\{ \begin{array}{l} \frac{36iu+102iu+48iu^6-36iu^7}{(1+u-u^6)^2} \\ +\frac{-36iu+42iu}{(1-u)^6}. \end{array} \right.$$

Now the fraction $\frac{-36iu+42iu}{(1-u)^6}$ is evidently equal to theſe two fractions, $\frac{6iu}{(1-u)^6} - \frac{42iu}{1-u^6}$,

the fluents of which are $\frac{6}{1-u} + 42 \times \text{h. l. } (1-u)$.

And, ſince $(1+u-uu)^6$ is the product of $(\frac{1+\sqrt{5}}{2}-u)^6 \times (\frac{1+\sqrt{5}}{2}+u)^6$, the other fraction on the right-hand ſide of the above equation may be reſolved into four fractions, of which the fluents will be two algebraic quantities and two logarithms.

* An English Translation of this valuable work was made by the late *Profeſſor* COLSON, of *Cambridge*, and just published from the Translator's Manuſcript, under the inſpection of the Rev. JOHN HELLINS, B. D. F. R. S.

But

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But an eaſier way of obtaining the fluent of the fraction $\frac{36u + 102uu + 48uu^2 - 36uu^3}{(1 + u - uu)^3}$ is, to aſſume it =

$$\frac{A + Bu}{1 + u - uu} + C \times \text{h. l.}\,(1 + u - uu) + D \times \text{h. l.}\,(\tfrac{1 + \sqrt{5}}{2} - u),$$

and to put the fluxion of this expreſſion = the given fluxion; from which the values of A, B, C, and D will be eaſily obtained; viz.

$$A = 21{\cdot}6, \qquad C = 0{\cdot}246315$$
$$B = 34{\cdot}8, \qquad D = -36{\cdot}492629$$

Now collecting theſe fluents together we have

$$\begin{cases} \frac{A + Bu}{1 + u - uu} + C \times \text{h. l.}\,(1 + u - uu) \\ + D \times \text{h. l.}\,(\frac{1 + \sqrt{5}}{2} - u) + \frac{6}{1 - u} + 42 \times \text{h. l.}\,(1 - u), \end{cases}$$

the ſum of which quantities, when $u = 0$, is

$A + D \times \text{h. l.}\,(\frac{1 + \sqrt{5}}{2}) + 6$. Therefore as x, y, and u vaniſh together, from the ſum of the fluents above found the conſtant quantities

$A + D \times \text{h. l.}\,(\frac{1 + \sqrt{5}}{2}) + 6$ muſt be ſubtracted and we ſhall have

$$y = \begin{cases} \frac{A + Bu}{1 + u - uu} + C \times \text{h. l.}\,(1 - u - uu) \\ + D \times \text{h. l.}\,(\frac{1 + \sqrt{5}}{1 + \sqrt{5} - 2u}) + \frac{6}{1 - u} \\ + 42 \times \text{h. l.}\,(1 - u) - A - 6; \end{cases}$$

in which expreſſion h. l. $(\frac{1 + \sqrt{5}}{1 + \sqrt{5} - 2u})$ is put for its equal $-$ h. l. $(\frac{1 + \sqrt{5}}{2} - u)$ + h. l. $(\frac{1 + \sqrt{5}}{2})$.

And the numerical values of the ſeveral terms, when $u = \sqrt{\frac{9}{10}}$ (if my calculation be correct,) are as below; viz.

A +

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$\frac{A+Bu}{1+u-uu} = 54{\cdot}85600$, $42 \times \text{h.l.}(1-u) = -170{\cdot}13756$,

$C \times \text{h.l.}(1+u-uu) = 0{\cdot}00418$, $-A-6 = -27{\cdot}6$

$-D \times \text{h.l.}(\frac{1+\sqrt{5}}{1+\sqrt{5}-2u}) = 34{\cdot}10775$, The sum $-197{\cdot}73756$

$\frac{6}{1-u} = 344{\cdot}69287$,

The sum $433{\cdot}66080$
$-197{\cdot}73756$

Diff. of the sums $235{\cdot}92324 = y$, which was required.

The Method of solving this question, as it was printed, by I. H.

Using the notation in the propofers solution, (which I have seen) we have

$\frac{6iu^5}{(1+2u^2+u^3)^2} = y$, of which fraction the denominator is the product of
$(0{\cdot}4533977 - 0{\cdot}2055694\,u + u^2)^2 \times (2{\cdot}2055694 + u)^2$;
and therefore, (putting $2{\cdot}2055604 = a$, for the sake of brevity,) by Art. 62 of the first section of the book above referred to, we may assume

$$\left.\begin{array}{l} \dfrac{Au + Biu + Ciu^2 + Diu^3}{(\frac{1}{a} - \frac{u}{aa} + uu)^2} \\ \quad + \dfrac{Eu + Fiu}{(a+u)^2} \end{array}\right\} = \frac{6iu^5}{(1+2u+u^3)^2},$$

from which equation the values of A, B, C, &c. will be had; and then, to find the fluent of

$\frac{Au + Biu + Ciu^2 + Diu^3}{(\frac{1}{a} - \frac{u}{aa} + uu)^2}$, (which is the only difficulty that remains), assume it $=$

$$\frac{G+Hu}{\frac{1}{a} - \frac{u}{aa} + uu} + I \times \text{h. l.}(\frac{1}{a} - \frac{u}{aa} + uu$$

$+ K$

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$+ K \times$ circ. arch whose rad. is $\frac{\sqrt{(4a^3-1)}}{2aa}$ and tang. $u - \frac{1}{2aa}$,

and put the fluxion of this expreſſion $=$ the given fluxion, by which means G, H, &c. will be determined. (See Art. 40, of the firſt ſection of the book before referred to).

The fraction $\frac{Eu + Fuu}{(a+u)^n}$ is $= \frac{(E - aF)u}{(a+u)^n} + \frac{Fu}{a+u}$

Now, collecting theſe fluents together, and correcting them, we have

$$y = \begin{cases} \dfrac{G + Hu}{\frac{1}{a} - \frac{u}{aa} + uu} - aG \\ + I \times \left\{ \text{h. l.} \left(\frac{1}{a} - \frac{u}{aa} + uu\right) + \text{h. l. } a \right\} \\ + K \times \left\{ \begin{matrix} \text{circ. arch of tang. } u - \frac{1}{2aa} \\ + \text{ circ. arch of tang. } \frac{1}{2aa} \end{matrix} \right\} \text{ and radius } \dfrac{\sqrt{(4a^3-1)}}{2aa} \\ + \dfrac{aF - E}{a+u} - F + F \times \left\{ \text{h. l. } (a+u) - \text{h. l. } a \right\}. \end{cases}$$

N. B. In this problem $\frac{\sqrt{(4a^3-1)}}{2aa}$ is $= \frac{\sqrt{(8aa+3)}}{2a}$.

The ſame, anſwered by Mr. Cunliffe.

Put $v^6 = x$, and the equation will become $\frac{6\dot{v}v^5}{(1-2v^6+v^{12})^n} = \dot{y}$.

Now it is pretty obvious that
$(1-v) \times (1+v-v^6) = 1-2v^6+v^7$: therefore aſſume
$\frac{Av+B}{(1-v)^n} + \frac{Cv^5+Dv^4+Ev+F}{(1+v-v^6)^n} = \frac{v^5}{(1-2v^6+v^{12})^n}$, by reducing the terms to a common denominator, &c. there will be had

$$\left.\begin{matrix} (A+C-1)v^5 + \left\{B + D - 2(A+C)\right\} v^4 \\ + \left\{E + C - A - 2(B+D)\right\} v^3 + (2A - B + F - 2E + D)v^2 \\ + (A + 2B - 2F + E)\, v + B + F \end{matrix}\right\} = 0.$$

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Making the coefficients of the homologous terms = o, we have, after proper reduction,
A = 7, B = — 6, C = — 6, D = 8, E = 17 and F = 6.
Wherefore,

$$\frac{Av+B}{(1-v)^3}+\frac{Cv^2+Dv^1+Ev+F}{(1+v-v^2)^3}=\frac{7v-6}{(1-v)^3}+\frac{6+17v+8v-6v^3}{(1+v-v^2)^3}.$$

Consequently the fluent of $\frac{v^6\dot{v}}{(1-2v^3+v^3)}$ $\Bigg\}$ $= \begin{cases} \text{fluent of } \frac{7v\dot{v}-6\dot{v}}{(1-v)^3} \\ +\text{flu. } \frac{6\dot{v}+17v\dot{v}+8v^2\dot{v}-6v^3\dot{v}}{(1+v-v^2)^3} \end{cases}$

$$= \begin{cases} \frac{1}{1-v}+7\times \text{h. l. } (1-v) \\ +\text{flu. } \frac{6\dot{v}+17v\dot{v}+8v^2\dot{v}-6v^3\dot{v}}{(1+v-v^2)^3} \end{cases}$$

Now $\frac{6\dot{v}+17v\dot{v}+8v^2\dot{v}-6v^3\dot{v}}{(1+v-v^2)^3}=\frac{6v\dot{v}-2\dot{v}}{1+v-v^2}+\frac{13v\dot{v}+8\dot{v}}{(1+v-v^2)^3}$ by $\div$.

But the denominators of these fractions will be more simple by taking away the second term; it will also be more obvious to which of the tabular forms the fluents of the several terms belong.

In order to take away the said second term of the denominators, put $z+\frac{1}{2}=v$, then will

$$\frac{6\dot{v}+17v\dot{v}+8v^2\dot{v}+8v^2\dot{v}-6v^3\dot{v}}{(1+v-v^2)^3}=\frac{6v\dot{v}-2\dot{v}}{1+v-v^2}+\frac{13v\dot{v}+8\dot{v}}{(1+v-v^2)^3}$$

$$=\frac{6z\dot{z}}{\frac{5}{4}-z^2}+\frac{\dot{z}}{\frac{5}{4}-z^2}+\frac{13\dot{z}\dot{z}+14\frac{1}{2}\dot{z}}{(\frac{5}{4}-z^2)^3}$$

$$=\frac{6z\dot{z}}{\frac{5}{4}-z^2}+\frac{13z\dot{z}+14\frac{1}{2}\dot{z}+\dot{z}(\frac{5}{4}-z^2)}{(\frac{5}{4}-z^2)^3}$$

$$=\frac{6z\dot{z}}{\frac{5}{4}-z^2}+\frac{15\frac{3}{4}\dot{z}+13z\dot{z}-z^2\dot{z}}{(\frac{5}{4}-z^2)^3}.$$

The fluent of the first term, viz. $\frac{6z\dot{z}}{\frac{5}{4}-z^2}$ is $-3\times$ h. l. $(\frac{5}{4}-z^2)$.

Assume $\frac{Mz+N}{\frac{5}{4}-z^2}+Q\times$ flu. $\frac{\dot{z}}{\frac{5}{4}-z^2}=$ flu. of $\frac{15\frac{3}{4}\dot{z}+13z\dot{z}-z^2\dot{z}}{(\frac{5}{4}-z^2)^3}$;

which expression put into fluxions is

$$\frac{M\dot{z}(\frac{5}{4}-z^2)+2z\dot{z}(Mz+N)}{(\frac{5}{4}-z^2)^3}+\frac{Q\dot{z}(\frac{5}{4}-z^2)}{(\frac{5}{4}-z^2)^3}=\frac{15\frac{3}{4}\dot{z}+13z\dot{z}-z^2\dot{z}}{(\frac{5}{4}-z^2)^3};$$

dividing by $\frac{\dot{z}}{(\frac{5}{4}-z^2)^3}$, and expanding the result there will

be

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be had $(M-Q)z^3+2Nz+\frac{5}{4}(M+Q)=15\frac{3}{4}+13z-z^3$, or $(M+1-Q)z^3+(2N-13)z+\frac{5}{4}(M+Q)-15\frac{3}{4}=0$.

Making the co-efficients of the homologous terms $=0$, there will be had, after proper reduction,

$M=\frac{29}{5}$, $N=\frac{13}{2}$ and $Q=\frac{34}{5}$. Therefore the fluent of

$$\frac{15\frac{3}{4}\dot{z}+13z\dot{z}-z^3\dot{z}}{(\frac{5}{4}-z^2)^2}=\frac{\frac{29}{5}z+\frac{13}{2}}{\frac{5}{4}-z^2}+\frac{34}{5}\times \text{flu.}\frac{\dot{z}}{\frac{5}{4}-z^2}$$

$$=\frac{\frac{29}{5}z+\frac{13}{2}}{\frac{5}{4}-z^2}+\frac{34\sqrt{5}}{25}\times \text{h. l.}\left(\frac{\frac{\sqrt{5}}{2}+z}{\frac{\sqrt{5}}{2}-z}\right),$$

because the fluent of $\frac{\dot{z}}{\frac{5}{4}-z^2}=\frac{1}{\sqrt{5}}\times \text{h. l.}\left(\frac{\frac{\sqrt{5}}{2}+z}{\frac{\sqrt{5}}{2}-z}\right)$.

Consequently the fluent of

$$\frac{6\dot{v}+17v\dot{v}+8v^2\dot{v}-6v^3\dot{v}}{(1+v-v^2)^2}=\begin{cases}\frac{\frac{29}{5}z+\frac{13}{2}}{\frac{5}{4}-z^2}+\frac{34\sqrt{5}}{25}\times \text{h. l.}\left(\frac{\frac{\sqrt{5}}{2}+z}{\frac{\sqrt{5}}{2}-z}\right)\\ -3\times \text{h. l.}\left(\frac{5}{4}-z^2\right)\end{cases}$$

$$=\begin{cases}\frac{29v+18}{5(1+v-v^2)}+\frac{34\sqrt{5}}{25}\times \text{h.l.}\left(\frac{\frac{\sqrt{5}-1}{2}+v}{\frac{\sqrt{5}+1}{2}-v}\right)\\ -3\times \text{h. l.}(1+v-v^2),\end{cases}$$

by restoring the value of z.

Therefore the fluent of

$$\frac{v^3\dot{v}}{(1-2v^2+v^3)^2}=\frac{1}{1-v}+7\times \text{h.l.}(1-v)+\text{flu.}\frac{6\dot{v}+17v\dot{v}+8v^2\dot{v}-6v^3\dot{v}}{(1+v-v^2)^2}$$

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$$= \begin{cases} \frac{1}{1-v} + 7 \times \text{h.l.}(1-v) + \frac{29v+18}{5(1+v-v^2)} \\ + \frac{34\sqrt{5}}{25} \times \text{h.l.} \left(\frac{\frac{\sqrt{5}-1}{2} + v}{\frac{\sqrt{5}+1}{2} - v} \right) - 3 \times \text{h.l.}(+v-v^2). \end{cases}$$

But by the queſtion this expreſſion ought to vaniſh when $v = 0$; therefore the correct fluent of

$$\left. \begin{array}{l} \frac{v^5 \dot{v}}{(1-2v^2+v^3)^2} \\ = \frac{y}{6} \end{array} \right\} = \begin{cases} \frac{1}{1-v} + 7 \times \text{h. l. } (1-v) + \frac{29v+18}{5(1+v-v^2)} \\ + \frac{34\sqrt{5}}{25} \times \text{h.l. } \frac{\frac{\sqrt{5}-1}{2} + v}{\frac{\sqrt{5}+1}{2} - v} + \text{h.l.} \left(\frac{\sqrt{5}+1}{\sqrt{5}-1} \right) \\ - 3 \times \text{ h. l. } (1+v-v^2) - \frac{23}{5} \end{cases}$$

$= 39{\cdot}3204703$, becauſe $v = \sqrt[6]{x} = \sqrt[6]{\frac{9}{10}} = {\cdot}982593175$.

Whence $y = 6 \times (39{\cdot}3204703) = 235{\cdot}9228218$.

W. W. R.

J. H. *is requeſted to ſend to Mr.* Glendinning's *for the Medal for ſolving the Prize Queſtion.*

ARTICLE

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ARTICLE XV.

MATHEMATICAL QUESTIONS.

To be anſwered in Number XV.

I. QUESTION 301, *by Mr.* William Francis, *Teacher of the Mathematics, Maidenhead, Berks.*

An octagon of fertile ſoil,
Whoſe copious bounds were juſt a mile,
Was let unto a baker
Who, fifty years, did pay no rent;
What can be due at *five per cent.**
And juſt *two pounds* per acre?

* *Compound Intereſt.*

II. QUESTION 302, *by Maſter* John Golding, *at Mr. Gregory's Academy, Cambridge.*

What is the axis of a globe when the ſuperfices and ſolidity are each expreſſed by the ſame number? 2nd. What is the ſuperficies when the axis and ſolidity are each expreſſed by the ſame number? 3d. What is the ſolidity when the axis and ſuperficies are each expreſſed by the ſame number?

III. QUESTION 303, *by Mr.* Francis.

Required the length of a very thin rectangular plate of braſs, 2 inches wide, that would exactly reach from the bottom diagonally acroſs to the top of a cylindrical ale quart, whoſe diameter is 4 inches.

IV. QUESTION 304, *by the late Mr.* John Watts, *of Aubourn, Wilts.*

Two poles were placed upright in the earth at 10 yards diſtance from each other to which a rope was faſtened 5 yards above the ground, but a heavy weight being fixed to the middle thereof depreſſed the rope two yards lower in that point and cauſed the poles to meet at the upper end. Query their length?

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V. QUESTION 305, *by Mr.* John Whitley.

Given $\left\{\begin{matrix} x + y = s\,(2) \\ x^9 + y^9 = b\,(32) \end{matrix}\right\}$ to find x and y by quadratics?

VI. QUESTION 306, *by Mr.* John Barron, *Spilsby.*

There is an air pump, the capacity of its receiver is to that of the barrel as 9 to 2. Now it is required to determine how many ſtrokes of the piſton will rarify the air in the receiver to the ſame denſity as the atmoſphere, at the altitude of 30 miles, when the denſity at the ſurface is 1?

VII. QUESTION 307, *by Mr.* Francis.

A gentleman having a quadrilateral garden ABCD, right-angled at A, wiſhes to render it more uniform by producing the ſide AD to E, which is alſo to be a right-angle, and ſo ſituated that the $\triangle$ DEC may be equal to the $\triangle$ ADB, formed by the two ſides AB, AD, and the diagonal line BD. Now the front door of his houſe, F, is equally diſtant from B and C, whoſe diſtance is the ſame, or 60 yards, from each other; moreover the lines BD, DC, and DF, are to each other as 9, 12, and 8 reſpectively. Hence the figure and area of the garden are required?

VIII. QUESTION 308, *by* Tyro Philomatheticus.

An elliptical field, the ſum of the diameters of which is 1000 yards, is to be ſold at half a guinea for every yard in length on the tranſverſe. and a guinea for every yard in length on the conjugate; required the dimenſions and area of the field, ſuppoſing it of the moſt advantageous form for the purchaſer?

IX. QUESTION 309, *by Mr.* Francis.

M. Grnerin's balloon appears to be compoſed of a cylinder, whoſe ends are covered with two hemiſpheres. Now its diameter being 24 and length 40 feet, what quantity of ſilk was required to make it? with what weight would it aſcend, admitting it to be filled with air 13 times lighter than atmoſpheric air, and what was the length of the parabolic tract deſcribed in his journey from Lord's ground, ſuppoſing his greateſt elevation to have been 3000 yards, and the diſtance of the places whence he aſcended and where he alighted 10 miles in a direct line?

X.

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X. QUESTION 310, *by Mr.* Thornoby.

Required the manner of valuing recurring ſurds continued *ad infinitum*, or to ſhew the manner of finding an equation by which their value may be found? and apply the method to the following

1ſt. $\sqrt{a+\cfrac{b}{\sqrt[n]{a+\cfrac{b}{\sqrt[n]{a+\cfrac{b}{\sqrt[n]{a+\cfrac{b}{\sqrt[n]{a+}}}}}}}}}$ &c.

2nd. $\sqrt{a+\cfrac{b}{\sqrt[n]{c+\cfrac{d}{\sqrt[m]{a+\cfrac{b}{\sqrt[n]{c+\cfrac{d}{\sqrt[m]{a+\cfrac{b}{\sqrt[n]{c+}}}}}}}}}}}$ &c.

3rd. $\sqrt{\cfrac{a}{b+\sqrt[n]{\cfrac{a}{b+\sqrt[n]{\cfrac{a}{b+\sqrt[n]{\cfrac{a}{b+\sqrt[n]{\cfrac{a}{b}}}}}}}}}}$ + &c.

XI. QUESTION 311, *by Mr.* Gregory, *Cambridge.*

There is a ſphere whoſe diameter is 12 inches; what is the ſide of its inſcribed cube; and what is the difference between the ſolidity of the greateſt inſcribed octagonal pyramid which can be inſcribed in the cube, and that of the greateſt circular ſpindle which can be inſcribed in either of the ſix ſpaces between the cube and the ſphere.

XII.

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XII. QUESTION 312, *by Mr.* William Peacock.

A Gentleman has a garden bounded by a ſemicircle FCD, and a ſemi-parabola FCAB, F being the vertex thereof. Now FB the abſciſſa of the parabola, (or diameter of the ſemicircle,) and AB the ordinate (rightly applied) being given, the former 100 and the latter 45 yards; it is required to find the length of the longeſt walk CD that can be made in the garden, parallel to AB.

XIII. QUESTION 313, *by Mr.* I. H. Swale.

B is a given point in the circumference of a circle given in magnitude and poſition, and P another given point, either within or without the circle; it is required to draw PAC, meeting the circle in A and C, ſo that if AB, CB be drawn, the ſum of their ſquares may be equal to a given ſpace.

XIV. QUESTION 314, *by Mr.* W. Roceſter.

Having given, as in queſtion 235 (vide No. X.), it is required to ſolve the problem when the ſum of the ſquares of the perpendiculars is equal to a given ſpace.

XV. QUESTION 315, *by* Limenus, *of Bruton, Somerſet.*

Let E be the centre of a circle inſcribed in the triangle ABC, which let meet the baſe AB in a; make Bb = Aa, and EM, drawn parallel to Cb will biſect the baſe in M. Required a demonſtration?

XVI. QUESTION 316, *by Mr.* William Ticken, *of the Royal Military College, Great Marlow, Bucks.*

At what part of the earth is the twilight abſolutely a minimum, on October 12th? and how much is the twilight ſhorter there than it is in the latitude 52° 12′ N. on the ſame day?

XVII. QUESTION 317, *by Mr.* Nicholas Bottom.

In a given triangle to inſcribe the greateſt ellipſis poſſible.

XVIII.

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XVIII. QUESTION 318, *by* Quid Nunc,

From Bonnycastles Algebra.

To find two whole numbers ſuch, that, if unity be added to either, or to their ſum or difference, the ſums thence ariſing ſhall be all ſquare numbers.

XIX. QUESTION 319, *by Mr.* Boſworth, *Cambridge.*

When the ſun ſhines upon a right cone ſtanding with its baſe upon a horizontal plane, the ſhadow of the cone will be bounded by lines drawn from the vertex of the ſhadow as tangents to the baſe of the cone. Required a demonſtration?

XX. QUESTION 320, *by Mr.* Johnſon, *Birmingham.*

Given the difference of the ſegments of the baſe made by the perpendicular, the ratio which the ſum of the ſquares of the ſides has to the rectangle contained by the perpendicular and a given line, to conſtruct the triangle when the ſum of the baſe and perpendicular is a maximum.

XXI. QUESTION 321, *by Mr.* John Whitley.

If Q be the point in the triangle, from which perpendiculars are drawn to the ſides, ſo that the ſolid contained under them is the greateſt poſſible; three times the ſum of the ſquares of the lines drawn from Q to the angular points of the triangle is equal to the ſum of the ſquares of the ſides of the triangle.

XXII. QUESTION 322, *by Mr.* I. F.

If ABCD be a circle given in magnitude and poſition, and if P be a given point without the ſame, a point Q may be found, within the circle, ſuch, that if any line whatever be drawn through P meeting the circle in C and D, and CQ, DQ be joined, the rectangle CQ × QD ſhall be equal to a certain given ſpace.

XXIII. QUESTION 323, *by Mr.* John Lowry.

Let ABC be a triangle inſcribed in the circle AFCBQ, whoſe centre is O, FDQ a diameter drawn perpendicular to the baſe, and meeting it in D, and let CE be drawn parallel to the baſe AB, and meeting DF in E. I ſay the ſolid contained by the ſquare of the

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the difference of the ſides, and the line drawn from the vertex to the middle of the baſe, will be a maximum, when the ſquare of DF (that part of the diameter above the baſe) is equal to thrice the rectangle DO × EF. Required the demonſtration?

XXIV. QUESTION 324, *by* Limenus, *Bruton.*

Join A, B, any two points in a conic ſection; biſect AB in C, and draw MN parallel to it, and terminated by the curve; draw MC*n* and NC*m* to the ſection, and the concourſe of M*m*, and N*n* will give the point P, from whence PA, PB, being drawn will touch the curve. Required a demonſtration?

XXV. QUESTION 325, *by Mr.* John Lowry.

Given the difference of the ſegments of the baſe, the exceſs of the ſquare of one ſide above the ſpace to which the ſquare of the other ſide has a given ratio, to conſtruct the triangle when the ſolid contained by the ſquare of the perpendicular and the baſe is a maximum.

XXVI. QUESTION 326, *by* Marlovienſis.

Required a ſimple finite expreſſion for the ſum of a ſlowly converging ſeries of this form,

$a + bx + cx^2 + dx^3 + ex^4 +$ &c. *ad infinitum*; in which all the terms are poſitive, or added to each other.

It is remarkable that this ſeries, which admits of perfect ſummation, is not to be found in any of the Analytic Writers, *De Moivre*, *Waring*, *Landen*, &c. Nor even in *Clarke's* Summation of Series; though the latter Treatiſe contains many forms that lead immediately to it.

XXVII. QUESTION 327, *by Mr.* James Cunliffe.

It is required to exhibit the whole area of the curve which is the *locus* of the middle point V, of the chain, independant of ſeries, from the coincidence of the points A and B, to the time that the chain becomes completely ſtretched along the horizontal line AB. See my ſolution to queſtion 265.

XXVIII.

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XXVIII. QUESTION 328, *by* Philalethes Cantabrigienſis.

If a ball of lead 10 inches in diameter were ſuſpended at a point by a very ſlender wire 20 inches in length, and made to vibrate in a vertical plane; and if, at the commencement of the motion, the wire being ſtretched out tight, ſhould be in a line parallel to the horizon, and the ball in ſuch a poſition that the wire, if continued, ſhould paſs through the centre of it; it is required to determine in what time the wire would firſt become perpendicular to the horizon?

XXIX. QUESTON 329, *by* Marlovienſis.

There is an uniform iron lever ACB, of the ſecond order, the weight of which is 2lb. avoird. *per* inch longitudinally. At C, 36 inches from the fulcrum A, is appended a weight of 100lb. Required what length the lever ought to be, that a power P, applied at the extremity B to raiſe the weight, may be the leaſt poſſible?

XXX. PRIZE QUESTION 330, *by* Marlovienſis.

In the common cycloid ABC, E and F are two points in the axis BD, equally diſtant from the middle: from theſe points lines are drawn parallel to the baſe, cutting the curve in K and L, The generating circle being deſcribed in the ſituation it would have when it generates the point K, and again when it generates the point L, call the points where theſe circles touch the baſe G and H. Now, the axis, and the points E, F, being given, it is required to exhibit, by a conciſe theorem, the area of the cycloidal ſpace contained within the arcs KG and LH without the aid of fluxions; and alſo to ſhew a geometrical method of cutting off a part from the generating circle equal to the ſame ſpace.

ARTICLE

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ARTICLE XVI.

Solution to Colonel Titus's Arithmetical Problem.

By James Ivory, *Esq.*

1. THIS problem requires to find the numbers a, b and c, from the following equations, viz.

$$\left.\begin{aligned} a^2 + bc &= 16 \\ b^2 + ac &= 17 \\ c^2 + ab &= 18 \end{aligned}\right\} \ldots \text{(No. 1.)}$$

2. Subtract the firſt of theſe equations from the ſecond, and the ſecond from the third, and

$$\left.\begin{aligned} b^2 - a^2 + ac - bc &= 1 \\ c^2 - b^2 + ab - ac &= 1 \end{aligned}\right\} \text{ or } \left\{\begin{aligned} (b-a) \times (b+a-c) &= 1 \\ (c-b) \times (c+b-a) &= 1 \end{aligned}\right.$$

$$\text{therefore } b + a - c = \frac{1}{b-a}$$

$$c + b - a = \frac{1}{c-b}$$

Put now, $b - a = m$, $c - b = n$, and we ſhall have

$$\left.\begin{aligned} b - a &= m \\ c - b &= n \\ b + a - c &= \frac{1}{m} \\ c + b - a &= \frac{1}{n} \end{aligned}\right\} \ldots \text{(No. 2.)}$$

3. From the equations (No. 2.) it is easy to get theſe values of a, b and c; viz.

$$\left.\begin{aligned} a &= n + \frac{1}{m} \\ b &= \tfrac{1}{2}\left(\frac{1}{m} + \frac{1}{n}\right) \\ c &= \frac{1}{n} - m \end{aligned}\right\} \ldots \text{(No. 3.)}$$

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In order to get an equation that shall contain only m and n and known numbers, I substitute the values of a, b and c, just found, in one of the original equations (No. 1.) as the first; and I get

$$(n + \tfrac{1}{m})^2 + \tfrac{1}{2}(\tfrac{1}{m} + \tfrac{1}{n}) \times (\tfrac{1}{n} - m) = 16.$$

We have still to find another equation that shall contain only m and n; but from the equations (No. 2) I get, $c - a = m + n$; and also $2c - 2a = \frac{1}{n} - \frac{1}{m}$: therefore $2m + 2n = \frac{1}{n} - \frac{1}{m}$.

We have thus two equations, which are sufficient to determine m and n; viz.

$$\left.\begin{aligned} (n + \tfrac{1}{m})^2 + \tfrac{1}{2}(\tfrac{1}{m} + \tfrac{1}{n}) \times (\tfrac{1}{n} - m) &= 16 \\ 2m + 2n &= \tfrac{1}{n} - \tfrac{1}{m} \end{aligned}\right\} \text{(No. 4)}$$

and m and n being obtained from these equations the numbers, a, b, and c that are sought will be had by means of the formulæ (No. 3).

4. It is not very easy (without a good deal of algebraic process) to deduce from the equations (No. 4) a final equation, containing only one unknown quantity: the shortest way of doing this appears to be the following.

$$(n + \tfrac{1}{m})^2 = (\tfrac{mn + 1}{m})^2 = \tfrac{(mn + 1)^2}{m^2}.$$

Again, $(\frac{1}{m} + \frac{1}{n}) \cdot (\frac{1}{n} - m) = (\frac{m + n}{mn}) \cdot (\frac{1 - mn}{n}) =$

$\frac{(m + n) \cdot (1 - mn)}{mn^2} = \frac{(m^2 + mn) \cdot (1 - mn)}{m^2n^2}$ (by multiplying both numerator and denominator by m).

Therefore, the first of the equations (No. 4) becomes, by substitution, $\frac{(1 + mn)^2}{m^2} + \frac{(m^2 + mn) \cdot (1 - mn)}{2m^2n^2} = 16$; and, putting $mn = x$, $m^2 = y$; $\frac{(1 + x)^2}{y} + \frac{(y + x)(1 - x)}{2x^2} = 16$.

Multiply both sides of the second of the equations (No. 4) by m^2n, and it becomes, $2m^2n + 2m^2n^2 = m^2 - mn$, or $2xy + 2x^2 = -x + y$.

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Thus we have these two equations, viz.

$$\left.\begin{array}{c} \frac{(1+x)^2}{y} + \frac{(y+x)(1-x)}{2x^2} = 16 \\ 2xy + 2x^2 = -x + y \end{array}\right\} \text{(No 5).}$$

In which $x = ms$, $y = m^2$.

5. It is now easy to deduce a final equation from the two equations (No. 5); because the unknown quantity y is only simply concerned, without any of its powers.

Multiply the first of the two equations by y, Then $(1+x)^2 + y(y+x).\frac{1-x}{2x^2} = 16y$: but from the second equation I get $y = \frac{x+2x^2}{1-2x}$; therefore $y + x = \frac{x+2x^2}{1-2x} + x = \frac{2x}{1-2x}$: therefore, by substitution

$$(1+x)^2 + \tfrac{1}{2}.\frac{x+2x^2}{1-2x} \times \frac{2x}{1-2x} \times \frac{1-x}{x^2} = 16\,\frac{x+2x^2}{1-2x}.$$

which is a final equation and only requires further to be properly reduced to prepare it for resolution.

Omitting the factors common to both numerator and denominator of the second member of the left-hand side, and multiplying all the terms by $(1-2x)^2$ I get

$$(1+x)^2(1-2x)^2 + (1+2x)(1-x) = 16(x+2x^2)(1-2x)$$

or

$$(1-x-2x^2)^2 + 1 + x - 2x^2 = 16x - 64x^3$$

or

$$2 - x - 5x^2 + 4x^3 + 4x^4 = 16x - 64x^3$$

or

$$2 - 17x - 5x^2 + 68x^3 + 4x^4 = 0$$

or, multiplying by 4 and transposing

$$8 = 34.2x + 5.4x^2 - 34.8x^3 - 16x^4.$$

And putting $2x = z$

$$8 = 34z + 5z^2 - 34z^3 - z^4.$$

6. Having determined z by the preceding equation we shall have $x = \frac{z}{2}$ and $y = \frac{x(1+2x)}{1-2x} = \frac{z}{2}.\frac{1+z}{1-z}$.

Also

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Alſo $mn=x$, $y=m^2$: therefore $m=\sqrt{(\frac{z}{2})}\times\sqrt{(\frac{1+z}{1-z})}$

and $n=\frac{x}{m}=\sqrt{(\frac{z}{2})}\times\sqrt{(\frac{1-z}{1+z})}$: Hence

$$a=n+\frac{1}{m}=\left\{\sqrt{(\tfrac{z}{2})}+\sqrt{(\tfrac{2}{z})}\right\}\times\sqrt{(\tfrac{1-z}{1+z})}=\tfrac{2-z-z^2}{2}\times\sqrt{(\tfrac{2}{z-z^2})}$$

$$b=\tfrac{1}{2}\left\{\tfrac{1}{m}+\tfrac{1}{n}\right\}=\tfrac{1}{2}\sqrt{\tfrac{2}{z}}\times\left\{\sqrt{(\tfrac{1+z}{1-z})}+\sqrt{\tfrac{1-z}{1+z}}\right\}=\sqrt{\tfrac{2}{z-z^2}}$$

$$c=\frac{1}{n}-m=\left\{\sqrt{\frac{2}{z}}-\sqrt{\frac{z}{2}}\right\}\times\sqrt{(\frac{1+z}{1-z})}=\frac{2+z-z^2}{2}\cdot\sqrt{(\frac{2}{z-z^2})}$$

or $b=\sqrt{(\frac{2}{z-z^2})}$

$$a=\frac{2-z-z^2}{2}\times b.$$

$$c=\frac{2+z-z^2}{2}\times b.$$

The Arithmetical Problem that is reſolved above was of ſome note in its day. It was firſt publiſhed by Dr. Wallis, in his Algebra, and it was propoſed to the Doctor by Colonel Silas Titus, a gentleman of the bedchamber to Charles the Second. Colonel Titus told Dr. Wallis, that the author of the problem was the famous Algebraiſt Dr. John Pell, who had propoſed it to the Colonel.

The Solution of the problem given by Dr. Wallis is remarkably operoſe and inellegant, Mr. Baron Maſeres has lately republiſhed Dr. Wallis's ſolution (in a Vol. of Tracts on the reſolution of Algebraic Equations, printed in 1800) and has been at great pains to ſet forth all the operations and reaſonings with his uſual clearneſs and accuracy. In the ſame vol. of Tracts there is likewiſe printed another ſolution of the ſame problem by Mr. Frend, which in point of ſimplicity is preferable to Dr. Wallis's ſolution.

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ARTICLE XVII.

Solution of a queſtion (the 16th, propoſed at page 69, Vol. I. but not anſwered) in this work, by Mr. JOHN FAREY.

To the Editor of the Mathematical Repoſitory.

SIR,

ON the appearance of the *Gentleman's Diary* in November 1789, I was much ſtruck with the aſſertion of the venerable and profound Mathematician who edited that work, ſupported by a process attempting to prove, that no part of the weight of a cylinder, circumſtanced as in your 16th queſtion Vol. I. p. 69, could be ſupported by the ſide of the veſſel; and I ſet about, and produced to my friends, *Mr George Sanderſon*, and others, an inveſtigation of the queſtion (609, *Gent's. Diary*) a copy of which you have below; but ſtill ſuſpecting the concluſions of my proceſs, oppoſed by ſuch high authority; I had recourſe to an experiment, the particulars of which you have alſo below. In a letter ſoon after to the *Rev.* Mr. WILDBORE, I objected to the principle laid down in the Diary for 1790, and recommended an experiment, and having the pleaſure of ſeeing that gentleman in Auguſt 1797, at his houſe in Nottinghamſhire, I re-ſtated my objections, which produced queſtion 753 in the Diary for 1798; accident however prevented my ſending a ſolution to that queſtion as I intended, including a ſolution alſo to queſt. 609. Before cloſing the preſent ſeries of your *Repoſitory* it may not therefore be improper to give a place to what follows, in answer to the 16th queſtion therein, which will oblige

Your Obedient Servant,

J. FAREY.

12, *Crown Street, Weſtminſter,*
26th December, 1803.

"On account of the equality of the cylindrical Hoofs, the calculation may without ſenſible error proceed, as though the cylindrical ſtaff was divided perpendicularly to the axis at the point of immerſion: it ſhould alſo be conſidered, that the buoyancy of the fluid will be exerted on the centre of gravity of the immerſed part, in a *vertical* direction, and ſince this forms one of the props or ſupports of the cylinder, the other ſupport i. e. the ſide of the veſſel muſt, to produce an equilibrium, act in a *vertical* direction alſo, and hence it is that the inclination of the ſtaff, or height of the ſide of the veſſel, is not a neceſſary datum, and that if a ſubſtitution be made for it, the ſame will fly off in reducing the equation,

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tion, and that the preſſure on the ſide, and the quantity or content of the ſtaff immerſed are conſtant quantities, not affected by the inclination of the ſtaff.

Fig. 655, Pl. 32. Let $AB = l$, repreſent the ſtaff, immerſed in the beer to C, and ſupported at A by a perpendicular prop; let D be the middle or centre of gravity of the ſtaff, F the middle or centre of gravity of the part without the beer, and E the middle or centre of gravity of the part immerſed. Put $AC = y$, $m =$ the weight of an inch in length of the ſtaff, and $n =$ the weight of an inch in length of the cylinder of beer diſplaced; then $AB \times m = lm$ is the weight of the ſtaff, $BC \times n$ is the weight of the beer diſplaced $= (l - y) \times n$, which exerts itſelf upwards at its centre of gravity E, and $DE = AC + CE - AD = y + \frac{l-y}{2} - \frac{1}{2}l = \frac{1}{2}y$. Now if a body at D be ſupported by two perpendicular props at A and E; it will be as DE $(\frac{1}{2}y)$: AD $(\frac{1}{2}l)$ :: preſſure at A : preſſure at E, and $(DE \times lm) \div (AD + DE) =$ the preſſure at $A = (\frac{1}{2}y \times lm) \div (\frac{1}{2}l + \frac{1}{2}y) = lmy \div (l + y)$; alſo $(AD \times lm) \div (AD + DE) =$ the preſſure at E, $= (\frac{1}{2}l \times lm) \div (\frac{1}{2}l + \frac{1}{2}y) = l^2m \div (l + y)$, this in the caſe of an equilibrium is equal to the action of the fluid, and $l^2m \div (l + y) = (l - y) \times n$, hence $l^2m = (l^2 - y^2) \times n = l^2n - y^2n$, and $ny^2 = l^2 \times (n - m)$, from whence we obtain $y = l\sqrt{(n-m) \div n}$ the length not immerſed, $m = n \times (l^2 - y^2) \div l^2$ the weight of an inch in length of the ſtaff, $n = m \times l^2 \div (l^2 - y^2)$ the weight of an inch in length of beer diſplaced, $l = y\sqrt{n \div (n - m)}$ the ſtaff's length, and by ſubſtituting the value of y in $lmy \div (l + y)$ we have $lm\sqrt{(n - m) \div n} \div (1 + \sqrt{(n - m) \div n}) =$ the preſſure on the ſide of the veſſel.

Now in the 16th queſtion $l = 36$ inches, $y = 13$; and the diameter of beer diſplaced being $\cdot 75$ inch, we have $\cdot 75 \times \cdot 75 \times \cdot 7854 = \cdot 4417875$ the content of an inch in length, which drawn into the weight of an inch of beer $\cdot 5949$, gives $\cdot 2628194$ oz. the weight of an inch in length of beer diſplaced $= n$, and $m = n \times (l^2 - y^2) \div l^2 = \cdot 228548$ oz. which drawn into 36 the length of the ſtaff, gives $lm = 8 \cdot 227719$ oz. the weight of the ſtaff as required. Moreover, $lmy \div (l + y) = 2 \cdot 182864$ oz. is the weight ſupported by the ſide of the veſſel, being in this caſe ſomething more than one fourth part of the whole weight of the ſtaff."

Note. Mr. WILDBORE makes a ſimilar obſervation to the laſt, at page 2 of the Diary for 1799, but it is proper to remark that the weight there given, is erroneous, becauſe derived on the

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ſuppoſition of the veſſel bearing no part of the weight. Alſo that $\frac{m}{n} = \cdot 8696$ is the ſpecific gravity of the ſtaff, beer being 1.

"*Experiment.* I procured a cylinder of light wood, which I painted, and when dry found its length 28·12 inches $= l$, its mean diameter 1·7 inches, and its weight 17·65 oz. whence $\frac{17 \cdot 65}{28 \cdot 12} = \cdot 627667$ oz. $= m$. I faſtened a thread to one end thereof, and attached it to one of the ſcales of a balance, ſuſpended by a pulley over a cistern of water; when the cylinder was immerſed and at reſt, I put 7·175 oz. into the other ſcale, which exactly balanced the pull of the thread; after which I lowered and again raiſed the ſcales, ſo that the cylinder passed into all degrees of elevation, and I had the pleaſure to obſerve the ſcales remain, as nearly as poſſible in equilibrium, and the thread vertical all the time. I alſo obſerved that about 19·5 inches of the cylinder's axis remained above the water at ſeveral elevations. When this cylinder was afterwards placed nicely upright in the water, and had ſunk to its equilibrium I obſerved that 13·98 inches were immerſed, by which if we divide the cylinder's weight 17·65 we get $n = 1 \cdot 262581$, and hence $y = l \sqrt{(n - m) \div n} = 19 \cdot 94$ inches, the exceſs of which above 19·5 inches was I believe in ſome part occaſioned by inaccuracy in meaſuring that length, but more by the inequalities of the cylinder, for on inverting it, the other end ſunk 14·45 inches into the water (inſtead of 13·98) which will give $n = 1 \cdot 221453$, and $y = 19 \cdot 61$, much nearer to 19·5 than the above. Further, $lny \div (l + y) = 7 \cdot 323$ oz. which differs but ·148 of an ounce from the actual tenſion of the thread."

P. S. 1ſt. In making the above experiment it was curious to obſerve, that when the cylinder was placed very nicely upright in the water, it ſunk till its whole weight was ſupported by the fluid, the depth immerſed being $ln \div n = 13 \cdot 98$ inches, when if the cylinder's axis was inclined but ever ſo little to the vertical line, it ſuddenly ſtarted up out of the water, till only $l - l \sqrt{(n - m) \div n}$ was immerſed $=$ about 8·18 inches, and the cylinder would have fell down, had not the thread ſupported it.

P. S. 2d. Notwithſtanding the weight ſupported by the thread was conſtant, in all the inclinations of the cylinder tried in the above experiment, yet it is certain, that when the cylinder has become nearly horizontal, the weight on the thread begins at a certain point to decreaſe, and continues ſo to do as the cylinder ſinks, till at length it floats on the water, and the thread ſuſtains no part

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part of its weight; some of the learned Correspondents to your *New Series of the Repository* will I hope give the investigation of these curious particulars.

J. F.

ARTICLE XVIII.

Demonstrations of Lawsons Propositions Proposed in

ARTICLE LII. VOL. II. P. 465.

PROP. XLVII. Fig. 603. Pl. 31.

Demonstrated by Messrs. Campbell, Dawes, *and* Johnson.

By similar triangles,

AD (CB) : ED :: BF : AB (DC), wherefore
CB·BF : ED·DC :: BF^2 : DC^2, and componendo,
CB·BF + ED·DC : ED·DC :: AF^2 : DC^2, But,
AE : AF :: DE : DC, consequently
EA·AF : ED·DC :: AF^2 . DC^2; therefore
EA·AF = ED·DC + CB·BF. *Q. E. D.*

The same, by Messrs. Lowry, Tyro Philomatheticus, Whalley, *and* Whitley.

Complete the rectangle BCEH, and draw BI perpendicular to AE. Then by similar triangles

AB (DC) : AI :: AE : DE, and
AE : EH (BC) :: BF : FI; therefore,
Rect. CDE = rect. EAI, and rect. CBF = rect. AE·FI; therefore
CDE + CBF = EAI + AE × FI = EAF.
Q. E. D.

PROP. XLVIII. Fig. 604, 605. Pl. 31.

Demonstrated by Messrs. Campbell, Dawes, Tyro Philomatheticus, *and* Whalley. Fig. 604.

Let ABC be any right-angled triangle, and DEBF a rectangle inscribed therein. By similar triangles

DE (BF) : FC :: AE : DF (EB), therefore
BFC : FC^2 :: AEB : EB^2, and perm. et comp.
AEB + BFC : AEB :: DC^2 : EB^2. But
AD : DC :: AE : DF, therefore,
ADC : DC^2 :: AEB : EB^2, or, permutando,

ADC

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ADC : AEB :: DC^2 : EB^2; therefore
AD·DC = AE·EB + BF·FC.

Q. E. D.

The ſame, by Meſſrs. Johnſon, Lowry, *and* Whitley. Fig. 605.

Let DBEF be a rectangle inſcribed in the right angled triangle ABC, and draw FS perpendicular to AC. Then by ſimilar triangles,

AD : AE :: ES : EF (DB), and
AE : ED (BF) :: FC : CS; therefore
Rect. ADB = AES, and rect. BFC = AE·CS; wherefore
ADB + BFC = AES + AE·CS = AEC.

Q. E. D.

PROP. XLIX. Fig. 606, 607, 608. Pl. 31.

Demonſtrated by Meſſrs. Campbell *and* Tyro Philomatheticus. Fig. 606.

Let HF be drawn and produced to meet AB in D, and CT, CG being the two tangents, from the centre O draw OT, OG; join TG, and let TC meet FH in I.

Becauſe the ∠s FTH, FGH are right ones, a circle will paſs through the points G, F, T, H; and the ∠ FTH will be equal to the ∠ OTI; therefore the ∠ ITH = OTB = OBT = AGT = FHT; conſequently the ∠ FIT = 2 ∠ FHT, and therefore I is the centre of the circle GFTH. In like manner it may be ſhewn, that GC meets FH in the ſame point; C and I therefore coincide. Moreover, the ∠ FHT = FGT = FBD, and the ∠ DFB = TFH; conſequently the ∠ BDF = FTH, = a right angle.

Q. E. D.

The ſame, by Meſſrs. Dawes, Whalley, *and* Whitley. Fig. 607.

Three perpendiculars form the angular points of a △ upon the oppoſite ſides, meet in the ſame point. In the △ ABH, AD and BE are perpendicular (Eu. 31, III.); therefore HG paſſing through F is perpendicular to AB. But it remains to be proved that C is in the line GH; draw EK perpendicular to AB, then the △s AEK, AHG being ſimilar, the ∠ AEK = AHG, but the ∠ IEA (CEH) = AEK (Eu. 32, III.); therefore the ∠ CEH = CHE, and the △ CHE is iſoſceles; in the ſame manner the △ CHD is proved to be iſoſceles; and the △s HEF, HDF being right angled at E and D, a circle will paſs through the points D, F, E, H, whoſe centre is in the common hypothenuſe HF, but CE = CH = CD; therefore C is the centre of the circle,

and

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and is in the line HG; consequently HC, CF produced will be perpendicular to AB.

Q. E. D.

The same, by Mr. John Lowry. Fig. 608.

Let G, D, be the points where the equal tangents touch the circle, and let the lines joining the points F, H meet the diameter in E; then because GB is perpendicular to AH, and AD to BH, HEF will be perpendicular to AB, by a well known property of plane △s; wherefore the truth of the propofition will appear manifeft if we can prove that two equal tangents drawn from the points D, G meet each other in the line HF. Now if they do meet in HF it muſt be where the radius OQ, drawn perpendicular to GD, meets HF, otherwiſe they could not be equal.

Let OQ produced meet HF at C and join CD, CG; then becauſe of the right angles at D and E, the points D, B, E, F are in a circle, therefore the ∠ DBG = DEC, but the ∠ DBG at the circumference on the whole arch DG, is = to the ∠ DOC at the centre, on half the arch DQ; therefore the ∠ DEC = DOC, and the points O, E, D, C are in a circle; wherefore the ∠ ODC = OEC = a right angle, therefore CD is a tangent to the circle at D; and in the ſame manner it may be ſhewn that CG is a tangent at G.

Since drawing up the above, I have been favoured with the following elegant demonſtration of this property by my ingenious and worthy friend Mr. John Fletcher. *I truſt he will excuſe the liberty I have taken in thus making it public.*

To the centre O draw DO, GO and join GD, DB; then ſince the angle GCD is the ſupplement of the angle GOD = the ſum of the angles AOG, BOD = twice the ſum of the angles ADG, BGD = twice the angle AFG, it follows that the angle GCD is equal to the angle at the centre of a circle paſſing through G, D, F; and as CG = CD, C is the centre itſelf, and therefore CF = CD; but the angle DBA is = CDA = CFD = AFE, and as A is common, AEF = ADB = a right angle. *Q. E. D.*

PROP. L. Fig. 609, Pl. 31.

Demonſtrated by Meſſrs. Campbell, Dawes, Johnſon, Lowry, Tyro Philomatheticus, Whalley, *and* Whitley.

Let BQ be the chord of 60°, equal to the radius, and join BF, FQ, and EF, then the angles BAF, BQF on the ſame arch are equal, and the triangles EAF, BFQ are iſoſceles; therefore they are ſimilar, and AF : EF :: EF (BQ) : FB

Q. E. D.

PROP.

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PROP. LI. Fig 610, Pl. 31.

Demonſtrated by Meſſrs. Campbell, Dawes, Johnſon, Lowry, Tyro Philomatheticus, Whalley, *and* Whitley.

Let the line AB be divided at C in extreme and mean proportion, that is, ſuch that AB : AC :: AC : CB then it is evident that AB^2 : AB × AC :: AB × AC : AB × CB, which is the proportion mentioned in the propoſition.

PROP. LII. Fig. 611, Pl. 31.

Demonſtrated by Meſſrs. Campbell, Dawes, Johnſon, Lowry, Tyro Philomatheticus, Whalley, *and* Whitley.

Let A, B, C, be the three lines. Then
$A : B :: B : C$, or $A \times C = B^2$, $\therefore A \times (A + C) = A^2 + B^2$.
Conſeq. $A^2 + B^2 : A \times B :: A \times (A + C) : A \times B :: A + C : B$.
Q. E. D.

PROP. LIII. Fig. 612, Pl. 31.

Demonſtrated by Meſſrs. Campbell, Dawes, Johnſon, Lowry, Tyro Philomatheticus, Whalley, *and* Whitley.

Let ABC be any right angled triangle, and CD perpendicular to the hypothenuſe AB.

The triangles ACB, CDB, being equiangular,
AC : AB :: CD : CB, or $2AC \times CB = 2AB \times CD$.
Therefore $(AC + CB)^2 = AC^2 + BC^2 + 2AC \times CB$,
$= AB^2 + 2AB \times CD$
$= AB \times (AB + 2CD)$.
Conſequently AB : AC + CB :: AC + CB : AB + 2CD.
Q. E. D.

PROP. LIV. Fig. 613, 614, 615, 616, 617, 618, Pl. 31.

Demonſtrated by Mr. Colin Campbell, *of Liverpool.* Fig. 613.

In AD produced take DG = AB, draw GE to meet the circle in K, and join AK, AE, KF. Becauſe
AB · BG = BE^2, AFK = AEK = a right ∠ = BAF (hyp.), therefore EK is parallel and equal to AD (= BG); therefore FB and KG are parallel; wherefore ∠ ABF = AGE = AEB, and therefore, ſeeing the △s AFB, ABE, BEG, are ſimilar,
AF : AB :: AB : BE :: BE : BG or AD. *Q. E. D.*

The

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The same, by Mr. John Dawes, *of Birmingham.* Fig. 614.

Continue AF to AL, make AL = BE; unite FB and LD, which, by hypothesis, must be parallel; to prove this, describe the semi-circles AHD, AMB, join AH, AE, HD, MB. Then it will be
AB : AH (BE) :: AH (AL) : AD, therefore the △s ABE, LAD are similar, and ∠ DAP = DLA, therefore the ∠ DPA is a right angle; draw HP, MR, and the ∠ HAP = HDP and MAR (HAP) = MBR; therefore HDP = MBR, and HD, MB are parallel, therefore LD and FB are parallel, and therefore AF : AB :: BE (AL) : AD. *Q. E. D.*

The same, by Mr. John Lowry. Fig. 615.

Produce EB to meet the circle again at S, and draw FI parallel to AB, meeting EB at I, then EI is evidently equal to BS, and by the property of the circle,

AB : BS (BE — AF) :: BE : BD; wherefore
AB × AD : AD × (BE — AF) :: BE² : BD × BE:
but AB × AD = BE², by hypothesis, therefore
AD × (BE — AF) = BD × BE, or
(AD — BD) × BE = AB × BE = AD × AF.
Wherefore AF : AB :: BE : AD. *Q. E. D.*

The same, by Tyro Philomatheticus, *of Hull.* Fig. 616.

Make DC = AB, ~~on AC describe a~~ semi-circle, then as BE is perpendicular to AC it will be a mean proportional between AB and BC, or AB and AD, and AF being perpendicular to AC, and FD drawn, it is manifest that FD is a diameter of the circle AFED; now the angles FED and AEC being each in a semi-circle, are right ones, and consequently equal; take away the common angle AED, and we shall have ∠ FEA = ∠ DEC, and (by Simpson's Geometry, III. 15.) the ∠ FAE = ∠ ECD; whence the △s FEA and DEC are similar, therefore

AF : CD = AB :: AE : CE, again, by sim. △s
AE : CE :: BE : BC = AD; whence
AF : AB :: BE : AD. *Q. E. D.*

The same, by Mr. James Whalley, *of Bolton.* Fig. 617.

On AD produced take DC = AB; then AD = BC; draw FP parallel to AD meeting BE in H, produce EB to meet the circle at G. Join AE, CE, and FB. Then

AB

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$AB \times BD = EB \times BG = EB \times EH$, and
AB : EB :: EH : BD (HP); but by constr.
AB : EB :: EB : BC, therefore
EH : HP :: EB : BC; wherefore CE passes through P, consequently $\angle BCE = HPE$, and arch $AG = FE$; hence $\angle AEB = FPE$, and $DC = AB$, $DP = AF$, therefore $FB = PC$, and the $\angle ABF = AEB$; the triangles ABF and ABE are therefore similar, consequently,
FA : AB :: AB : BE :: BE : BC or AD. *Q. E. D.*

The same, by Mr. John Whitley, *of Attercliffe Academy.*
Fig. 618.

Join DE and through F, draw EFG meeting DA produced in G. Then because the $\angle DAF$ is a right angle, the $\angle DEF$ will also be a right angle, therefore
$DB \times BG = BE^2$, but $AD \times AB = BE^2$, therefore
$AB \times AD = DB \times BG$; therefore
AD : DB :: BG : AB, and by division
AB : DB :: AG : AB; and by hypothesis
AF : AB :: BE : AD, also,
AB : BE :: BE : AD, therefore
AF : AB :: AB : BE, and
$AF \times BE = AB^2$, therefore
AG : AF :: BE : DB; which is true, the triangles AGF, DBE, being evidently similar.

ARTICLE XIX.

Demonstrations of Lawson's Propositions proposed in ARTICLE VI. VOL. III. P. 71.

PROP. LV. Fig. 618, Pl. 31.

Demonstrated by Messrs. Campbell, Collins, Dawes, Johnson, Tyro Philomatheticus, Whalley, *and* Whitley.

Let ACB be a triangle right-angled at C, and CD perpendicular to the hypothenuse AB. Then
AB : AC :: AC : AD, and convertendo,
AB : AB — AC :: AC : AC — AD, and permutando
AB : AC :: AB — AC : AC — AD, wherefore
AB — AC : AC — AD :: AC : AD. *Q. E. D.*
PROP.

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PROP. LVI. Fig. 619, 620, 621, Pl. 31.

Demonſtrated by Meſſrs. Campbell, Johnſon, *and* Whalley. Fig. 619.

Draw the radius OC; then by ſimilar triangles,

KH : HC :: HC : HO, therefore
KH·HO = HC² = BH·HD; conſequently
BH : DH :: BK : KD (Conv. Prop. I.), therefore
FH : HG :: FI : IG (Prop. II.); hence
FH² : FH·FG = HC² :: FI : IG. *Q. E. D.*

The ſame, by Meſſrs. Dawes *and* Whitley. Fig. 620.

Upon FG as a diameter deſcribe the ſemi-circle GLF, join *p*P, CF, CG; P, *p* being the centre of the circle and ſemi-circle; draw the tangent LH, which is equal to CH becauſe their ſquares are each = to GH·HF; and join LG, LF and LI; LI is perpendicular to GF. For the △s HKI and H*p*P are ſimilar; therefore PH·HK = *p*H·HI; but conceive CP, L*p* to be joined then PH·HK = HC² and therefore LI is perpendicular to FG. Now the △s HFC and HCG being ſimilar, as alſo HFL and HLG.

HF² : HC² :: FC² : GC², and
HF² : HI² (HC²) :: LH² : LG². Therefore
FL² : LG² :: FC² : CG²; but
FL² : LG² :: FI : IG; therefore
FC² : GC² :: FI : IG; and therefore
FH² : HC² :: FI : IG. *Q. E. D.*

The ſame, by Meſſrs. Tyro Philomatheticus *and* Collins. Fig. 621.

From the centre O draw OE ⊥ to FG and join OC, then by ſimilar triangles,

HO : HE :: HI : HK therefore
HE·HI = HO·HK = HC² = HG·HF; whence
HG : HE :: HI : HF, or
2HG : 2HE = HG + HF :: HI = HF + FI : HF, and
2HG − (HG + HF) = FI + IG : FI :: HF + FG : HF;
wherefore by diviſion and converſion
HF : HG :: FI : IG, and therefore
HF² : HG × HF = HC² :: FI : IG. *Q. E. D.*

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PROP. LVII. Fig. 622, 623, 624, 625, Pl. 31.

Demonſtrated by Mr. Colin Campbell. Fig. 622.

Draw BG ⊥ and BF ‖ to AC, and O being the centre of the circumſcribing circle, join OC.

The angle BAC = to the angle ACB, therefore
ABG = CBG = OCB (becauſe O is in BG), therefore
2OCB = ABC = ACB = CBE + CEB = 2CEB; co ſeq.
GCO = CEB = FBD; alſo FB = AG = OC;
Therefore OC is equal to DB. *Q. E. D.*

The ſame, by Meſſrs. Collins, Dawes, Whalley, *and* Whitley. Fig. 623.

Draw the perpendiculars BI, CK, (which continue to H); their interſection, G, is the centre of the circle circumſcribing the triangle, but AC = CE,

Therefore HC is parallel to DE, but AI = IC, therefore HG = GC, but BG = GC, therefore BGHD is an equilateral parallelogram, and BD = BG. *Q. E. D.*

The ſame, by Mr. Johnſon, *Birmingham.* Fig. 624.

To the centre of the circumſcribing circle O, draw AO, BO, and DO.

Now the ∠s CBE + ABC = a right angle,
therefore the ∠s DBO + DAO = 2 right angles.

Hence, by Eu. 22d. III. a circle will paſs through the points D, B, O, A, therefore in the triangles DBO, BAO,
the ∠ BAO = ABO = BDO = DOB.
But AO = OB, therefore OB = DB. *Q. E. D.*

COROL. The right line DO biſects the ∠ ADB.

The ſame, by Tyro Philomatheticus. Fig. 625.

Deſcribe a circle about the Δ and draw BF. Then
∠ DFB = BCA = 60°, and CEB + CBE = BCA;
but CE being equal to CB, ∠ CBE = CEB = 30°,
therefore the complement of CEB = ADE = 60°;
Whence BD = BF, and FAC being a right angle,
FBC = a rt. ∠, but BAC = ABC, ∴ FAB = FBA.

Conſe-

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Conſequently BF = FA = the ſide of a regular hexagon, which, it is well known is equal to the radius of the circumſcribing circle. *Q. E. D.*

PROP. LVIII. Fig. 626, Pl. 31.

Demonſtrated by Meſſrs. Campbell, Collins, Dawes, Johnſon, Tyro Philomatheticus, Whalley, *and* Whitley.

Join CE. Becauſe ∠ ADB = CEB, the triangles ABD, CBE are ſimilar, therefore

BE : BD :: CB : AB; but
DC : AD :: CB : AB, (Euc. VI. 3.); conſequently
EE : BD :: DC : AD. *Q. E. D.*

PROP. LIX. Fig. 627, Pl. 31.

Demonſtrated by Meſſrs. Campbell, Collins, Dawes, Johnſon, Tyro Philomatheticus, Whalley, *and* Whitley.

Because BA : BD :: BD : Be, and BC : BD :: BD : BE, the △s ABD, DBe; and CBD, DBE, are ſimilar; whence ∠ AED = ∠ ADB, and ∠ DeC = ∠ CDB; and therefore by ſimilar triangles
AB : AD :: AD : AE, and CB : CD :: CD : Ce.
Therefore BAE = AD² and BCE = CD². *Q. E. D.*

PROP. LX. Fig. 628, Pl. 31.

Demonſtrated by Meſſrs. Campbell, Collins, Dawes, Johnſon, Tyro Philomatheticus, Whalley, *and* Whitley.

Becauſe the vertical ∠ of the triangle ABC, inſcribed in the ſemi-circle ABC is a right angle, the triangles AFE, EGC are ſimilar to ABC, and conſequently to one another.

Therefore AE : EF :: GE : EC,
therefore GE·EF = AE·EC = EH².
Conſequently EG : EH :: EH : EF. *Q. E. D.*

From the preceding demonſtration it appears that the triangle need not have been reſtricted to an iſoſceles one.

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ARTICLE XX.

Anſwers to the Mathematical Queſtions propoſed in ARTICLE V. No. XII.

I. QUESTION 271, *anſwered by* Yanto.

The owners ſtock at the beginning of any year may be eaſily found by means of a table of the different orders of the figurate numbers. Thus at the beginning of

the 1ſt year his ſtock will be	1
2d	1
3d	1
4th	1 + 1
5th	1 + 2
6th	1 + 3
7th	1 + 4 + 1
8th	1 + 5 + 3
9th	1 + 6 + 6
10th	1 + 7 + 10 + 1
11th	1 + 8 + 15 + 4
&c.	&c. &c. &c.

Hence it appears that the owners ſtock at the beginning of the 31ſt year will be equal to the ſum of

the 31ſt term of the ſeries	1. 1. 1 &c.	= 1
28th	1. 2. 3 &c.	= 28
25th	1. 3. 6 &c.	= 325
22d	1. 4. 10 &c.	= 2024
19th	1. 5. 15 &c.	= 7315
16th	1. 6. 21 &c.	= 15504
13th	1. 7. 28 &c.	= 18564
10th	1. 8. 36 &c.	= 11440
7th	1. 9. 45 &c.	= 3003
4th	1. 10. 55 &c.	= 220
1ſt	1. 11. 66 &c.	= 1
That is, the owners whole ſtock is		= 58425

In like manner all queſtions of this nature may be very readily anſwered.

Suppoſe

Suppoſe the number of years, to be 20, inſtead of 30. Then it is evident that the owners whole ſtock will be equal to the ſum of

the 21ſt term of the ſeries	1.	1.	1	&c
18th	1.	2	3	&c
15th	1.	3.	6	&c
12th	1.	4.	10	&c
9th	1.	5.	15	&c
6th	1.	6.	21	&c
3rd	1.	7.	28	&c.

The ſum in this caſe will be 1278.

Again, ſuppoſe the number of years to be 40 inſtead of 30, and that the calves begin to breed at the end of every two years.

In this caſe the owners whole ſtock at the beginning of the 41ſt year will be equal to the ſum of

the 41ſt term of the ſeries	1.	1.	1	&c
39th	1.	2.	3	&c
37th	1.	3.	6	&c
35th	1.	4.	10	&c
&c.			&c.	

The ſtock in this caſe is found to be 165580141.

The ſame, anſwered by Mr. James Campbell, *at Mr. Flower's Academy, Iſlington.*

By conſidering the nature of this queſtion. I obſerve that the number of cows that calved at the end of theſe years will be as follows, viz.

	3.	4.	5.	6.	7.	8.	9.	10.	11.	12	.....
	1.	1.	1.	2.	3.	4.	6.	9.	13.	19	.....

and ſo on to 27. 28. 29. 30 years.

5896. 8641. 12664. and 18560, reſpectively, which are found by adding the laſt to the laſt but two.

Let $s =$ the whole ſum, and w, x, y, and z the four laſt terms of the ſeries; then

$1+1+1+2+3+4+6+9+13....+w+x+y+z=s$, and

$1+1+1+2+3+\&c.....+w=s-x-y-z$;

Take the latter of theſe from the former, and we have

$1+1+1+1+2+3+4+6+$ &c..... $+x=x+y+z$;

add z to both ſides of this equation and we get

$1+1+1+1+1+2+3+4+6+$ &c..... $+x+z=x+y+2z$,

which is evidently the owners whole ſtock of cows and calves at the end of any number of years, and in the preſent caſe is equal to $8641+12664+2\times 18560=58425$. *W. W. R.*

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The ſame, by Mr. William Francis, *Taplow Academy.*

Let the lines in fig. 629. pl. 31. repreſent the years to the 10th, and the dots thereon the calves produced in their reſpective years. It will then appear, that the produce of calves, in any year, added to the laſt year but one, will give the number that will be produced the following year. Hence the number produced at the commencement of each year will be as follows, viz.

1ſt	2d	3d	4th	5th	6th	7th	8th	9th	10th	11th	12th
1.	0.	0.	1.	1.	1.	2.	3.	4.	6.	9.	13.

13th	14th		29th	30th	31ſt
19.	28	and ſo on to	8641.	12664	18560.

The ſum of which is 58425 the number required.

Solutions to this queſtion were alſo received from Meſſrs. Boole *and* Collins.

II. QUESTION 272, *anſwered by Mr.* Francis.

Let ABCD (fig. 630, pl. 31.) repreſent a ſection of the tub through its axis, the line gB the tranſverſe diameter of the elliptical ſurface of the water when ſtanding upon the inclined plane.

Draw BE perpendicular and gh parallel to CD, interſecting each other in n. Put $CD = a$, $AB = b$, $BE = h$, and $gh = x$.

Then $ED = \frac{1}{2}(a - b)$, $nh = \frac{1}{2}(x - b)$, and $gn = \frac{1}{2}(x + b)$; and by reaſon of the parallels ED, nh,

$$ED. : EB :: nh : nB = \frac{EB \cdot nh}{ED} = \frac{h(x - b)}{a - b}.$$

And by corol. 2d Prob. 26th Sect. 1ſt of Hutton's Menſuration.

$$\frac{gh\sqrt{(gh \cdot AB) - AB^2}}{gh - AB} \times .2618\,Bn \times AB, \text{ or}$$

$(x\sqrt{[bx]} - b^2) \times \frac{.2618\,hb}{a - b}$ will expreſs the content of the ungula gAB, which put $= c$, the cubic inches 2 ale gallons; Whence $x\sqrt{[bx]} = \frac{c(a - b)}{.2618hb} + b^2$, and $x = 21.4115889$.

And by the principles of trigonometry,

$$\frac{Pn}{gn} = \frac{2h(x - b)}{(a - b) \times (x + b)} = .170434, \text{ nat. tang. of } \angle Bgh,$$

the

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the inclination of the plane upon which the tub ſtands. From whence, by help of the beſt tables, the $\angle Bgh$ is found to be 9° 40′½ nearly.

And thus the queſtion was anſwered by Meſſrs. Cunliffe, Marrat, Thornoby, *and* Whitley.

III. QUESTION 273, *anſwered by the Propoſer.*

To avoid complex Fractions, put $12x$ = the diameter of the cone's baſe, 12 being diviſible both by 3 and 4; then, by the rules of menſuration

$$144x^2 \times \frac{3{\cdot}141593}{4} \times \frac{\text{alt.}}{3} = 2412{\cdot}743424,$$ the ſolidity;

whence we find the altitude $= \frac{64}{x^2}$, and conſequently the ſlant height $= \sqrt{(36x^2 + \frac{4096}{x^4})}$; this multiplied into $6x \times 3{\cdot}141593$ half the circumference of the baſe gives the curve ſurface, that is

$$18{\cdot}849558x\sqrt{(36x^2 + \frac{4096}{x^4})} = 753{\cdot}98232.$$ Which equation reduced gives $x = 2$: wherefore the diameter of the baſe is 24, the altitude 16, and the ſlant height 20.

Now with reſpect to a conic ungula formed by a vertical plane; it was firſt ſhewn by Father Guido Grandi, an Italian Geometrician, and has been well demonſtrated by Mr. Emerſon in the Ladies' Diary for 1738*, that the ſegment of the baſe formed by the vertical plane is to the curve ſurface of the ungula, as the radius of the baſe, to the ſlant height of the cone. Find therefore, the verſed ſine of that ſegment of the baſe which contains ⅓ of it; thus, from a table of circular ſegments the verſed ſine of a ſegment containing ⅓ of the area of a circle to the diameter 1 is ·254069; therefore

As 1 : 24 :: ·254069 : 6·097656 = AS (fig. 681, pl. 31.).

Then AP — AS = 12 — 6·097656 = 5·902344 = PS;

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* An ingenious inveſtigation of a ſimilar theorem was given by Mr. Cunliffe in No. XIII. *Repoſitory*, page 109. See alſo Dr Hutton's tranſlation of Montucla's Ozanam, Vol. I. page 416.

and

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and laſtly, as is manifeſt from the conſideration of Mr. Emerſon's Theorem, and the nature of ſimilar triangles,
As AP (12) : AB (20) :: SP (5·9023 &c.) : 9·83724 = VB, the diſtance required from the vertex of the cone.

Neat ſolutions to this queſtion were alſo received from Meſſrs. Cunliffe, Francis, Thornoby, *and* Whitley.

IV. QUESTION 274, *anſwered by Mr.* Thomas Bazley.

The weight p ſuſpended on the wheel BFCD (fig. 632, pl. 31.) elevates the weight q ſuſpended on the axle AEC; it is required to determine the preſſure on the axis C.

Through the centre C and the points of contact A and B, draw DACB. Put CA $= a$, CB $= b$; then it is plain by the property of the lever, that $\frac{qa}{b}$ is the weight which hung at B would balance q at A; and therefore $p - \frac{qa}{b} = \frac{pb - qa}{b}$ is the motive force acting at B*. Now the maſs moved reduced to the diſtance CD or CB is (Sect. 6. Prop. I. Atwood on Motion) $\frac{qa^2 + pb^2}{b^2}$;

whence the accelerative force is

$$\frac{pb - qa}{b} \div \frac{qa^2 + pb^2}{b^2} = \frac{b(pb - qa)}{qa^2 + pb^2}.$$

Again, the weight q reduced to the diſtance CD $=$ CB, by the Prop. above quoted, is $\frac{qa^2}{b^2}$, and conſequently

$\frac{qa^2}{b^2} + \frac{qa}{b}$ is the maſs cauſing the tenſion of the ſtring pB (Sect. 10.

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* The enſuing part of the ſolution will appear more perſpicuous, by conceiving (as in fig. 633, pl. 31.) the weight q to be diſperſed (uniformly) in the periphery of the circle AEC, and the weight $\frac{qa}{b}$ diſperſed in the periphery of the circle BFDC; and the weight $W = p - \frac{qa}{b}$ as giving motion to this ſyſtem.

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10. Prop. 1. Ibid.); and therefore that tenſion is

$$\left(\frac{qa^2}{b^2} + \frac{qa}{b}\right) \times \frac{b\,(pb - qa)}{qa^2 + pb^2}$$

that is, the maſs into the accelerating force. Laſtly, add to this, the weight of the ſyſtem, viz. $q + \frac{qa}{b}$, and we have the whole preſſure on the axis

$$= \left(q + \frac{qa}{b}\right) + \left(\frac{qa^2}{b^2} + \frac{qa}{b}\right) \times \frac{b\,(pb - qa)}{qa^2 + pb^2},$$

which is eaſily reduced to $\frac{pq\,(a+b)^2}{qa^2 + pb^2}$.

Corol. If $b = a$, as is the caſe in the pulley we have $\frac{pq \times (2a)^2}{(q+b)a^2} = \frac{4pq}{p+q}$ for the preſſure on the axis.

Hence whatever be the radius of the pulley, the preſſure on the axis is the ſame.

Meſſrs. Marrat *and* Thornoby *alſo ſent ſolutions to this queſtion.*

V. QUESTION 275, *anſwered by Mr.* Cunliffe.

Fig. 634. Pl. 31. From the centre O upon DM let fall the perpendicular On, and join OF. Put AB = BD = $2a$, BM = x, and MF = MP = y, then MO = $x - a$ and by Euc. I. 47. DM = $\sqrt{(4a^2 + x^2)}$: and by reaſon of the ſimilar $\triangle$s MDB, MOn,

$$\mathrm{M}n = \frac{x\,(x-a)}{\sqrt{(4a^2 + x^2)}} \text{ and } \mathrm{O}n = \frac{2a\,(x-a)}{\sqrt{(4a^2 + x^2)}}.$$

$$\text{Again } n\mathrm{F}^2 = \mathrm{OF}^2 - \mathrm{O}n^2 = \frac{a^2 \times (8ax - 3x^2)}{4a^2 + x^2};$$

$$\text{Whence } n\mathrm{F} = a\sqrt{\left(\frac{8ax - 3x^2}{4a^2 + x^2}\right)}, \text{ and therefore}$$

$$\mathrm{MF} = \mathrm{MP} = \mathrm{M}n + n\mathrm{F}, \text{ that is,}$$

$y = \frac{x(x-a)}{\sqrt{(4a^2 + x^2)}} + a\sqrt{\left(\frac{8ax - 3x^2}{4a^2 + x^2}\right)}$, is an equation expreſſing the nature of the curve which is the locus of the point P. Moreover

$y\dot{y} = \frac{x\dot{x}\,(x-a)}{\sqrt{(4a^2 + x^2)}} + a\dot{x}\sqrt{\left(\frac{8ax - 3x^2}{4a^2 + x^2}\right)}$ is the fluxion of the

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the area. The correct fluent of the firſt member is

$$\frac{x}{2}\sqrt{(4a^2+x^2)}-a\sqrt{(4a^2+x^2)}+2a^2-2a^2\times \text{h.l.}\left(\frac{x+\sqrt{(4a^2+x^2}}{2a}\right).$$

But the fluent of $ax\sqrt{\left(\frac{8ax-3x^2}{4a^2+x^2}\right)}$ is not to be expreſſed by any method which I am acquainted with, without infinite ſeries; it may be integrated by means of a ſeries as follows. Aſſume

$$\sqrt{\left(\frac{8ax-3x^2}{4a^2+x^2}\right)}=Ax^{\frac{1}{2}}+Bx^{\frac{3}{2}}+Cx^{\frac{5}{2}}+Dx^{\frac{7}{2}}+\&c.$$

By ſquaring the equation, and expanding $\frac{8ax-3x^2}{4a^2+x^2}$ by diviſion; and then by equating the homologous terms on each ſide of the equation, we ſhall obtain

$$\sqrt{\left(\frac{8ax-3x^2}{4a^2+x^2}\right)}=$$

$$\frac{\sqrt{(2a)}}{2a}\times:\left(2x^{\frac{1}{2}}-\frac{3x^{\frac{3}{2}}}{8a}-\frac{73x^{\frac{5}{2}}}{256a^2}+\frac{165x^{\frac{7}{2}}}{4096a^3}+\frac{13085x^{\frac{9}{2}}}{262144a^4}\&c.\right)$$

Therefore the fluent of $ax\sqrt{\left(\frac{8ax-3x^2}{4a^2+x^2}\right)}=$

$$\sqrt{(2ax)}\times:\left(\frac{2x}{3}-\frac{3x^2}{40a}-\frac{73x^3}{1792a^2}+\frac{165x^4}{36864a^3}+\frac{13035x^5}{2883584a^4}\&c.\right)$$

Wherefore the correct expreſſion for the ſpace DMP, is

$$\frac{x}{2}\sqrt{(4a^2+x^2)}-a\sqrt{(4a^2+x^2)}+2a^2-2a\times\text{h.l.}\left(\frac{x+\sqrt{(4a^2+x^2)}}{2a}\right)$$

$$+\sqrt{2ax}\times:\left(\frac{2x}{3}-\frac{3x^2}{40a}-\frac{73x^3}{1792a^2}+\frac{165x^4}{36864a^3}+\&c.\right.$$

This Queſtion was ingeniouſly anſwered by Meſſrs. Marrat *and* Whitley.

VI. QUESTION 276, *anſwered by Mr.* George Roy.

Since the pendulum compoſed of the ball of copper, and a ſlender rod of given length l in inches, and given weight w ounces, made v vibrations in s ſeconds, it will be eaſy to find the diſtance from the point of ſuſpenſion to the centre of oſcillation of the compound pendulum. For $\frac{v}{s}$ will expreſs the number of vibra-

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tions in 1 second; and its reciprocal $\frac{s}{v}$ = the time of one vibration. And as the distances from the points of suspension to the centres of oscillation of pendulums, are as the squares of the times of vibration,

we have $1^2 : \frac{s^2}{v^2} :: 39\frac{1}{8} : \frac{39\frac{1}{8}s^2}{v^2} =$ PV in fig. 635. pl. 31. V being the centre of oscillation of the compound body. Here v and s being given, PV becomes known, and may, therefore, be denoted by L. Now in the figure, PB being $= l$, if C be the centre of gravity of the rod, and O its centre of oscillation, then will PC $= \frac{1}{2} l$ and PO $= \frac{2}{3} l$. Call BI the radius of the ball, x; then, conceiving it suspended at B, and F its centre of oscillation, we shall have IF $= \frac{2}{5} x$. And if c = weight in ounces of a cubic inch of copper, and $n = 5236 \times 8$; cnx^3, will represent the weight of the ball. Now M denoting the mass of the cylinder, and M′ that of the sphere, we have, by the nature of the centre of oscillation,

$$\text{PV} = \frac{(\text{PO} \times \text{PC} \times \text{M}) + (\text{PF} \times \text{PI} \times \text{M}')}{(\text{PC} \times \text{M}) + (\text{PI} \times \text{M}')}.$$

Let the respective lines and masses in this expression be denoted as above, and it will become

$$\text{L} = \frac{(\frac{1}{2} l \times \frac{2}{3} l \times w) + (l + x) \times (l + \frac{7}{5} x) \times cnx^3}{\frac{1}{2} lw + (l + x)\, cnx^3}$$

This, after a little reduction, becomes

$$\left.\begin{matrix} \frac{7}{5} cnx^5 + \frac{1}{2} cln \\ - cLn \end{matrix}\right\} x^4 + \left.\begin{matrix} l^2 cn \\ - Lcn \end{matrix}\right\} x^3 = \tfrac{1}{2} Llw - \tfrac{1}{3} l^2 w:$$

an equation which it is not necessary to throw into any other form, until some values of L, l, w, &c. are specified; when x may readily be found, and consequently cnx^3 the weight of the copper ball.

VII. QUESTION 277, *answered by Mr.* Cunliffe.

Let AB (fig. 636, pl. 31.) represent the cylinder just immersed in the fluid, with its axis perpendicular to the surface. By the principles of hydrostatics the pressure of the fluid upon any part Ea of the surface of the cylinder is equal to the weight of a column of the fluid, whose base is equal to the surface of the part, and height equal to $\frac{1}{2}$(Ba).

That

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That is, the preſſure upon the ſurface of the part B*a*, is proportional to the ſolid content of a priſm, whoſe baſe is equal to the ſurface of the part B*a*, and height $\frac{1}{2}$(B*a*): But the ſurface of the part B*a* is as the height B*a*, therefore the preſſure upon the ſurface of the part B*a* is as $(Ba)^2$, or as the ſquare of the height B*a*. And conſequently the preſſure upon the ſurface of the part A*a*, will be as $(BA)^2 - (Ba)^2$. And as the preſſures upon theſe two parts are to be equal by the queſtion, we ſhall have

$$(BA)^2 - (Ba)^2 = (Ba)^2, \text{ or } (BA)^2 = 2(Ba)^2;$$

wherefore if B*a* be taken upon BA equal to the ſide of a ſquare whoſe diagonal is BA, *a* will be the point where the cylinder is to be cut, ſo that the preſſures upon the ſurfaces of the parts B*a* and *a*A ſhall be equal to each other.

Et cætris paribus, if it was required to cut the cylinder, ſo that the preſſures upon the ſurfaces of the two parts might have the given ratio of *m* to *n*; that is, that the preſſure upon the convex ſurface of the upper part (B*a*) may be to the preſſure upon the convex ſurface of the lower part (*a*A) as *m* to *n*. Then from what has been deduced

$$(Ba)^2 : (BA)^2 - (Ba)^2 :: m : n, \text{ whence}$$

$(Ba)^2 = \frac{m\,(BA)^2}{m+n}$ which will determine the point *a* where the cylinder is to be cut in this caſe.

The ſame, anſwered by Mr. William Marrat, *of Boſton.*

Put BA the length of the cylinder $= l$, and let the diſtance of the ſection from the ſurface of the water, or $BA = x$; then, becauſe the preſſure at A is as $AB = AC$, the preſſure at any point *a* will be as B*a* or *ab*; therefore the preſſure on the part B*a* will be as the area of the $\triangle$ *ab*B, and that againſt *a*A as the area C*ba*A; that *is*, $\frac{1}{2}x^2 = \frac{l+x}{2} \times (l-x) = \frac{1}{2}(l^2 - x^2)$, by the queſtion.

Therefore $x = l\sqrt{(\frac{1}{2})}$.

This queſtion was likewiſe ingeniouſly anſwered by Meſſrs. Francis, Thornoby, *and* Whitley.

VIII. QUESTION 278, *anſwered by* Ambulator.

After the body had paſſed over the plane AB, (fig. 670. pl. 32.) if there were no loſs of velocity at the angle B, the time of running over the ſpace $BD = 2AB$ with the acquired velocity, would be equal

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equal to the time employed in passing down AB; and, in this case, since the times of running down planes of equal altitudes are as their lengths, the problem would be solved by simply setting off from A to BD, $AE = 2AB$.——But, the whole velocity at B : the velocity along BD :: AB : BC :: aB : Bc :: rad. : cos. B. Hence, since the times are inversely as the velocities, if T be the time of running over BD with the whole velocity acquired in falling down AB, we have cos. B : rad. :: T : $\frac{T}{\text{cos. B}}$ time of running over BD with the reduced velocity. Or, if AB represent the time down AB, we have cos. B : rad. :: AB : $\frac{AB}{\text{cos. B}}$ :: AB : $\frac{AB^2}{BC}$, for the time along BD. Hence take $AE = AB + \frac{AB}{\text{cos. B}}$ $= AB + \frac{AB^2}{CB}$, and set from A to BD, and AE is the length and position of the required plane.

It was also answered by Mr. Marrat.

IX. QUESTION 279, *answered by Mr.* Cunliffe.

Let QCP (fig. 638. pl. 31.) be the given circle, O its centre; and imagine the bodies A and B to move at the same time from the point P, the body A along the periphery, and the body B along the given chord PQ, in such a manner that the straight line AB connecting them may always pass through the centre O of the circle. It is very obvious that the curve which is the locus of the centre of gravity of the two equal bodies A and B, or which is the same thing the middle point in the straight line connecting them will pass through the points P and Q. Draw the radius OmC perpendicular to PQ, and cutting it in m, and let QGP represent the curve which is the locus of the centre of gravity of the two bodies, and let V be the point where this curve cuts the radius OC, and draw GH perpendicular to OC. Put $CV = Vm = c$, $Om = b$, $VH = x$, and $HG = y$.

Then $mH = c - x$, and $OH = b + c - x$; And by Euc. I. 47. $OG = \sqrt{(OH^2 + HG^2)} = \sqrt{\{(b+c-x)^2 + y^2\}}$

And by reason of the parallels mB, HG;

OH : mH :: OG : BG = GA, and by composition

OH : Om + 2mH :: OG : OG + GA = OA = OC,

 whence

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whence $(Om + 2mH) \times OG = OH \times OC$,
which being expreſſed algebraically is
$\left\{b+\frac{1}{2}2(c-x)\right\} \sqrt{\left\{(b+c-x)^2+y^2\right\}} = (b+c-x)\times(b+2c)$, or
$\sqrt{\left\{(b+c-x)^2+y^2\right\}} = \frac{(b+c-x)\times(b+2c)}{b+2(c-x)}$, which
is therefore the equation of the curve which is the locus of the centre of gravity of the two bodies.

When $2(c-x) = -b$, that is $2(x-c) = b$, or $x = c + \frac{1}{2}b$, it is plain that y will be infinite. Hence, it appears that the curve has two infinite legs; and the aſſymptote is parallel to the given chord QP, and biſects Om.

X. QUESTION 280, *anſwered by Mr.* Cunliffe.

Let DIB (fig. 639, pl. 31.) be half the harmonical curve, CD its axis, and CB its greateſt ordinate. Suppoſe the rectangle HIKC to be the greateſt that can be inſcribed in the curve; and with the radius CD and centre C deſcribe a circle cutting the ordinate HI in F, and CB in Q. Put $DC = r$, $DH = x$, $HI = y$, the arc $DF = z$, and $c = 3{\cdot}1416$, half the circumference of a circle whoſe radius is 1. Then by the property of the curve $y = \frac{az}{r}$, where a is a conſtant quantity eaſily determined from the data. Whence $HI \times CH = \frac{az}{r} \times (r-x)$ will expreſs the area of the rectangle CI, which is to be a maximum by the queſtion; therefore $z \times (r-x)$ muſt be a maximum; putting its fluxion $= 0$, $r\dot{z} - z\dot{x} - x\dot{z} = 0$, whence $z\dot{x} = (r-x) \times \dot{z}$; but it is well known that $\dot{z} = \frac{r\dot{x}}{\sqrt{(2rx-x^2)}}$, by means of which, exterminating $\dot{z}$ out of the preceding equation, and it will become
$z\dot{x} = \frac{r\dot{x}(r-x)}{\sqrt{(2rx-x^2)}}$; whence $z = \frac{r(r-x)}{\sqrt{(2rx-x^2)}}$;
or the arc $DF = \frac{CD \times CH}{HF}$. Now $\frac{CD \times CH}{HF}$ is known to expreſs the tangent of the arc QF, or the cotangent of the arc DF. Wherefore the queſtion is now to find ſuch an arc of a circle as ſhall be equal to its cotangent. An arc of a circle equal to its cotangent,

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tangent, from a few approximations by the method of trial and error will be found to be 49° 17′ ⅗ very near. The cosine of 49° 17′ ⅗ to rad. 1 is ·652092, and therefore CH = ·652092 × r = 1·304184. Alſo, by the property of the curve and the queſtion CB = $(a \div r) \times$ quadrant DQ = ½a × 3·1416 = a × 1·5708 = 10; whence a = 10 ÷ 1·5708. Moreover, the length of the arc

DF = 1·7206659; whence HI = $(a \div r) \times$ DF = $\frac{5}{1{\cdot}57^{c}8}$ × 1·7206659 = 5·477037, and 2HI = 10·954074 will be the whole length of the greateſt rectangle that can be inſcribed in the harmonical curve whoſe baſe is 20, and height 2, and from what has been before deduced CH = 1·304184 the height of the rectangle; Hence the area is found to be 14·286128.

The ſame, by Mr. John Whitley, *Attercliffe.*

Let MHQGN (fig. 640, pl. 31.) repreſent the harmonic curve, and ſuppoſe that CHGD is the required rectangle. Put OI = OF = a = 6·366618, OQ = Om = b = 2, and x = OP. Then by the property of the circle IF = $\sqrt{\{(a+x) \times 2a\}}$, EF = $2\sqrt{(a^2 - x^2)}$, and the length of the arc

EIF = $\frac{8\sqrt{\{(a-x) \times 2a\}} - 2\sqrt{(a^2-x^2)}}{3}$ =, by the property of the curve, to GH. And by ſim. △s,

OF : OP :: Om : On = $\frac{bx}{a}$; therefore

GH × CH = $\frac{8\sqrt{\{(a-x) \times 2a\}} - 2\sqrt{(a^2-x^2)}}{3} \times \frac{bx}{a}$ = a max. per queſtion; the fluxion of which being made = o, and reduced gives

$(4a - 6x)\sqrt{\{(a-x) \times 2a\}} = a^2 - 3x^2$; from this equation x is found = 4·1589 nearly, and the area of the rectangle = 14·268.

Mr. George Roy *and* Mr. Thornoby *anſwered it.*

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XI. QUESTION 281, *answered by the Proposer*, Mr. Dawes.

In fig. 641, pl. 31. let ASL represent the required spherical triangle, make the $\angle$ CSL = half the given $\angle$ ASL, continue SL to G so that SG = 91° 39′ = half the given sum. Draw the great circle GC perpendicular to SG and let it meet SC in C; CL making the $\angle$ CLG = half the complement of the given $\angle$ ALS.

By Trigonometry

As tang. GLC : tang. GSC :: fin. GS : fin. GL = 51° 41′ 29″.

Hence LS = 39° 57′ 31″, therefore AS = 80° 1′ 26″,

and LA = 63° 19′ 3″, which were required.

XII. QUESTION 282, *answered by* Cap. Geo. Gorry.

It has often been shewn that a whole number, having no integral root, has no terminating decimal root, and it is generally admitted, as stated in the question, that the decimal obtained by extracting any root of a whole number cannot be a repetend; but I know of no investigation of this matter which is so perspicuous, and at the same time so concise, as the one given in Vol. II. of Mr. Manning's Introduction to Arithmetic and Algebra. It may be found at page 84, of that ingenious performance, and is as follows:

"If possible, let N be a number the figures of whose mth root circulate; and let $\frac{S}{R}$ be equal to the value of the previous figures, and the value of the circulating part. I say in that case

$$\sqrt[m]{N} = \frac{S}{R}, \text{ or } N = \frac{S^m}{R^m}.$$

For, if N be not equal to $\frac{S^m}{R^m}$, let

$$N = \frac{S^m}{R^m} \pm d = \frac{S^m \pm dR^m}{R^m}; \text{ then } \sqrt[m]{N} = \frac{\sqrt[m]{(S^m \pm dR^m)}}{R};$$

That is, the figures obtained for $\sqrt[m]{N}$ agree throughout with the decimal fraction obtained by dividing the figures of $\sqrt[m]{(S^m \pm dR^m)}$ by R; but the figures of $\sqrt[m]{N}$ by hypothesis agree with those obtained by

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by dividing S by R, therefore the figures of $\sqrt[m]{(S^m \pm dR^m)}$ are the figures of S, or $\sqrt[m]{(S^m \pm dR^m)} = S$, which is abſurd; therefore in that caſe $\sqrt[m]{N}$ would be equal to $\frac{S}{R}$; but becauſe, by hypotheſis, N has no integral root, neither has it any fractional root; Therefore, the decimal, obtained by the operation of extracting the root of a whole number cannot circulate."

Mr. Cunliffe *ſent an anſwer to this queſtion, and ſo did Mr.* Whitley.

XIII. QUESTION 283, *anſwered by Mr.* Marrat.

ANALYSIS. (Fig. 642, pl. 31.) Let ABC repreſent the triangle with a circle inſcribed, L being the point of contact thereof with the baſe AB. Through the centre O draw CO to meet the baſe in I, alſo draw CK perpendicular to AB, and let E be the middle of AB. By the queſtion LK is given, and EL is known to be equal to half the difference of the ſides AC, BC, which is given by the queſtion. Alſo by Prop. IV. of Harriſon's Propoſitions in Vol. 1ſt of the Repoſitory $EI \times EK = EL^2$, whence EI becomes known, that is, the point I is given. Hence the following conſtruction.

In the ſame right line take EL = ½ the given diff. of the ſides, and LK = the diff. of the ſegs. of the baſe made by the point of contact of the inſcribed circle, and that of thoſe made by the perpend. and draw KC perpend. to EK. Now the point I being determined by the Analyſis, from I to KC, apply IC = to the given line biſecting the vertical angle. Draw LO perpend. to EK meeting IC in O; then with the centre O and radius OL deſcribe a circle, and from the point C draw two right lines to touch the circle and meet EK in A and B, and ABC is the triangle required, as is evident from the Analyſis.

Meſsrs. Cunliffe, Dawes, *and* Swale *anſwered it.*

XIV. QUESTION 284, *anſwered by* Quinbus Fleſtrin.

Let CAB (fig. 643, pl. 31.) repreſent a rectangular quadrant of a given ellipſis, E the point of contact of the inſcribed circle with the curve, join CE, and draw ED and EF at right angles to

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CA and CB respectively. Also draw GH to touch the curve in E, and meet CA, CB, in G and H respectively. Put $CA = a$, $CB = c$, $CD = EF = x$, and $DE = CF = y$.

By Euc. I. 47. $CE^2 = CD^2 + DE^2 = x^2 + y^2$, and by the known property of the ellipsis $c^2 \times (a^2 - x^2) = a^2y^2$.

Also $CD \times CG = CA^2$, or $CD = CA^2 \div CG = a^2 \div x$;

and $CF \times CH = CB^2$, or $CH = CB^2 \div CF = c^2 \div y$.

Moreover in the right angled triangle GCH, E is the point of contact of the inscribed circle with the hypothenuse GH, therefore by a well known property of the rectangle

$GE \times EH = \frac{1}{2}(CG \times CH) = a^2c^2 \div 2xy$.

And by Lawson's Theorems, Prop. 36, and what has been deduced above

$CD \times CG + CF \times CH = CE^2 + GE \times EH = CA^2 + CE^2$, that is $x^2 + y^2 + a^2c^2 \div 2xy = a^2 + c^2$; from this equation and from the property of the ellipse quoted above, x and y may be determined, and by means of these the radius SE easily found.

The same, by Mr. Paffman, *the Proposer.*

Fig. 644, pl. 31. BNKD is the given quadrant of the ellipsis. Let NST represent the greatest inscribed circle, touching the ellipsis in N. Draw NC and OT perpend. to BD; and NM and OS perpend. to LD. AL is a tangent to the ellipsis and circle at the point N. Let $a = BD$, $c = DK$, $x = DC$.

Then, by Emerson's Conic Sections $\frac{a^2}{x} = AD$,

and, by the same $CN = \frac{a}{c}\sqrt{(a^2 - x^2)}$.

Moreover $AC = \frac{a^2}{x} - x = \frac{a^2 - x^2}{x}$, and by Eu. I. 47.

$NA = TA = \sqrt{\left\{\left(\frac{a^2 - x^2}{x}\right)^2 + \frac{c^2}{a^2} \times (a^2 - x^2)\right\}}$; conseq.

$DT = \frac{a^2}{x} - \sqrt{\left\{\left(\frac{a^2 - x^2}{x}\right)^2 + \frac{c^2}{a^2} \times (a^2 - x^2)\right\}}$, the radius of the inscribed circle. By sim. $\triangle$s,

$AC : AN :: CD\ (NM) : NL\ (LS) = x^2\sqrt{\left(\frac{1}{x^2} + \frac{c^2}{a^4 - a^2x^2}\right)}$.

And

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And as AC : CN :: AD : DL = $\frac{ac}{\sqrt{(a^2-x^2)}}$.

Hence DS = $\frac{ac}{\sqrt{(a^2-x^2)}} - x^2\sqrt{(\frac{1}{x^2} + \frac{c^2}{a^4-a^2x^2})}$.

But DS is equal to DT, that is,

$$\frac{a^2}{x} - (\frac{a^2-x^2}{x})^2 + \frac{c^2}{a^2} \times (a^2-x^2) =$$

$$\frac{ac}{\sqrt{(a^2-x^2)}} - x^2\sqrt{(\frac{1}{x} + \frac{c^2}{a^4-a^2x^2})}. \quad \text{Or,}$$

$$\frac{a^2}{x}\sqrt{(a^2-x^2)} + (2x^2-a^2)\sqrt{(\frac{a^2-x^2}{x^2} + \frac{c^2}{a^2})} = ac.$$

From this equation x may be found.

The same, by Mr. W. Wallace, *of the R. M. College.*

Analysis. Let *ale* (fig 650, pl. 32.) be the given elliptic quadrant. Suppose the problem resolved, and that the inscribed circle touches the ellipse at *d* and the axes at *e* and *f*. Let ABC be a similar elliptic quadrant, in which there is inscribed a circle DEF of a given magnitude; then, if CA, CB, the semi-axes of this ellipse, be found, it is evident that *e*, *f* the points at which the circle touches the axes of the given ellipse may also be found; for, since the circle DEF is given in magnitude, the lines CE, CF are given in magnitude, each being equal to its radius, therefore, if the semi-axes CA, CB be determined, they will be divided at E and F into given segments; but the elliptic quadrants *abc*, ABC being similar, the semi-axes *cb*, CB will be similarly divided at *f* and F, therefore the ratio of *cf* to *fb* is given, and *cb* being given the the point *f* may be found. In like manner it will appear that the point *e* may be found; but when the points at which the circle touches the axes are determined, the method of inscribing the circle is obvious. We proceed to investigate the semi-axes AC, BC.

Let G be the centre of the circle EDF. Join DG cutting the axes in H and K, then DHK will be a normal to the curve, therefore by a known proposition in Conics, KH is to HD as the square of the excentricity to the square of the semi-transverse axis, which ratio is given, seeing it is the same as that of the like quantities in the given ellipse *abc*. In GD take L so that GK may be to GL in the same given ratio of HK to HD; then HK : HD :: HG

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:: HG : GL — HD; moreover the point G, and the line CK being both given by position the point L is in a ſtraight line given by poſition; let this line be NQ.

Take GM = DL, then GL = DM, and GL — HD = DM — HD = HM; hence the proportion HK : HD :: HG : GL — HD becomes HK : HD :: HG : HM; now the ratio of HK to HD being given, as already ſhewn, the ratio of HG to HM is alſo given; but the point G and the line CH are both given by poſition, therefore the point M is in a ſtraight line given by poſition: Let this line be NR.

We have now found that DG paſſes through a given point G and meets two lines NQ, NR, both given by poſition, in L and M, ſo that DL = GM, hence it appears that the *Locus* of the point D is a hyperbola which paſſes through G, and has NQ, NR for its aſymptotes; but D is alſo in the circumference of a given circle, therefore the point D, and the poſition of the normal DG are both determined.

Through D draw a perpendicular to the normal meeting the axes produced in S and T, this line, which is evidently a tangent to the curve at D, will be given by poſition, therefore the points S, T are both given. Draw DX, DY perpendiculars to the axis, and the points X, Y will alſo be evidently given. Now, from the nature of the ellipſe, the ſemi-tranſverſe CB is a mean proportional between CT and CY, which lines have been ſhewn to be both given in a magnitude, therefore the ſemi-tranſverſe axis is alſo given in magnitude. In like manner it appears that the ſemi-conjugate axis CA, which is a mean proportional between CS and CX, is given in magnitude, but the ſemi-axes AC, BC being once found we have already ſhewn that the whole of the ſolution is obvious.

Solutions to this queſtion were alſo received from Meſſrs. Geo. Roy, Marrat, *and* Whitley.

XV. QUESTION 285, *anſwered by Mr.* Swale.

Fig. 646, pl. 91. The baſe and vertical angle being given, the circumſcribing circle from thence becomes known; let this circle be deſcribed, in which apply the chord AB equal to the given baſe of the triangle, draw the diameter FH to biſect AB in E, and join FB. Upon FH produced take FN equal to the given aggregate of the ſum of the ſides and line biſecting the vertical angle: then produce NF to R in ſuch a manner that FR × RN =

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$= FB \times (AB + FB)$; to AB apply $FL = FR$, which produce till it cuts the circle in C, join AC, BC, and ABC is the triangle required.

DEMONSTRATION. The triangles ACL and FLB are similar, and CL bisects the angle ACB,

therefore $FL : FB :: AB : AC + BC$,

whence $FL \times (AC + BC) = FB \times AB$;

Also $FL : FB :: FB : FC$,

whence $FL \times FC = FB^2$, and by adding these together

$FL \times (AC + BC + FC) = FB \times (AB + FB)$.

But by the constr. $FR \times RN = FL \times RN = FB \times (AB + FB)$.

Consequently $RN = AC + BC + FB$, and by taking away $FR = FL$, there will remain $FN = AC + BC + LC$, the given aggregate of the sum of the sides and line bisecting the vertical angle; therefore the triangle ABC is that required.

Thus it was answered by Messrs. Cunliffe *and* Whitley.

XVI. QUESTION 286, *answered.*

Mr. Gregory, the proposer, informs us that the case exhibited for consideration in this question is the same as one of those described by Dr. Desaguliers, in the Philosophical Transactions, No. 407, or abridgment Vol. VI. p. 307. He also finds that the same case is considered in S. Clark's Mechanics, and some later Treatises on that subject, a circumstance of which he was not aware, at the time he proposed the question.

The solution is deduced from very obvious principles. For the oblique force DP (fig. 534, pl. 28.) is resolvable into two others, one as CD perpendicular to the horizon, and the other equal to CP, and parallel to the horizon; this latter force is also parallel to the beam FC which has no tendency to turn it round its centre of suspension F. The force DC then is the only effective force, and must, because action and re-action are equal, be considered both acting *upwards* at the point P, and downwards at the point D or C. Therefore the effect of the oblique thrust DP will be represented by $(CF \times CD) - (FP \times CD)$, and this effect will be shewn in the motion of the scale D downwards.

Now, rad. : sine $\angle$ DPC :: DP : DC;

that is 1 : ·8660254 :: 8 : 6·9282032 stones perpendicular force. And

$(CF - PF) \times CD = 6 \times 6{\cdot}9282032 = 4{\cdot}15692192$ stones, additional weight, which must be added to W in order to counterbalance the oblique thrust DP.

Had

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Had P been beyond C; with reſpect to F the fulcrum, then an oblique thruſt, as DP, would have puſhed the arm FC upwards; and this would be counter-balanced by taking away from the weight W, a weight proportional to (FP — FC) × CD.

Mr. Marrat *alſo favoured us with a ſolution to this queſtion.*

XVII. QUESTION 287, *anſwered by Mr.* Cunliffe.

Let P and r denote the ſame things as in my ſolution to Queſtion 287; and n the time required.

Then after the firſt quarterly payment the debt will be denoted by

$$Pr - 1,$$

after the ſecond by $Pr^2 - r - 1$,

after the third by $Pr^3 - r^2 - 2r - 3$,

after the fourth by $Pr^4 - r^3 - 2r^2 - 3r - 4$;

and therefore after the nth quarterly payment the debt will be denoted by

$$Pr^n - r^{n-1} - 2r^{n-2} - 3r^{n-3} - 4r^{n-4} \text{ &c. } - n$$

$$= r^n \times \left\{ P - \left(\frac{1}{r} + \frac{2}{r^2} + \frac{3}{r^3} + \frac{4}{r^4} + \text{&c. } \frac{n}{r^n} \right) \right\}.$$

And by the ſaid ſolution

$$\frac{1}{r} + \frac{2}{r^2} + \frac{3}{r^3} + \frac{4}{r^4} + \text{&c. } + \frac{n}{r^n} = \frac{r^{n+1} + n - (n+1)r}{r^n \times (r-1)^2};$$

Wherefore $r^n \times \left\{ P - \left(\frac{1}{r} + \frac{2}{r^2} + \frac{3}{r^3} + \frac{4}{r^4} + \text{&c. } \frac{n}{r^n} \right) \right\}$

$$= Pr^n - \frac{r^{n+1} + n - (n+1)r}{(r-1)^2}$$

$$= \frac{Pr^{n+2} - (2P+1) \times r^{n+1} + Pr^n + (n+1) \times r - n}{(r-1)^2}$$

$$= \frac{r^n \times \left\{ Pr^2 - (2P+1)r + P \right\} + (n+1) \times r - n}{(r-1)^2}.$$

And this expreſſion is to be a maximum by the queſtion,

or

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or $r^n \times \left\{ Pr^{n} - (2P + 1)\, r + P \right\} + (n + 1) \times r - n$

is to be a maximum; wherefore putting its fluxion $= o$, and dividing by $\dot{n}$, there will be had

$$\left\{ Pr^{n-2} - (2P + 1) \times r^{n+1} + P r^{n} \right\} \times \text{h.l.}(r) + r - 1 = o,$$

which when properly reduced gives

$$r^n = \frac{r - 1}{\left\{ r - P \times (r - 1)^2 \right\} \times \text{h. l. } (r)} = g,$$

whence again $n = \text{h. l. } (g) \div \text{h. l. } (r) = 12{\cdot}70726.$

XVIII. QUESTION 288, *anſwered by Mr.* Cunliffe.

Let a denote the diameter of the circle, whoſe chord is l, and verſed ſine h; and c the diameter of the circle, whoſe chord is b, and verſed ſine h. Alſo let x denote the perpendicular diſtance of any ſection of the groin parallel to the baſe from the vertex. The ſides of the ſaid ſection will be denoted by $2\sqrt{(ax - x^2)}$ and $2\sqrt{(cx - x^2)}$.

And from the nature of the figure and what is done at Art. 165. Vol. 1ſt. Simpſon's Fluxions,

$$\left.\begin{aligned} &4\sqrt{(ax - x^2)} \times \frac{\frac{1}{2}a\dot{x}}{\sqrt{(ax - x^2)}} \\ &+ 4\sqrt{(cx - x^2)} \times \frac{\frac{1}{2}c\dot{x}}{\sqrt{(cx - x^2)}} \end{aligned}\right\} = 2a\dot{x} + 2c\dot{x}, \text{ is the fluxion}$$

of the convex ſurface; the fluent of which is $2ax + 2cx = 2x\,(a + c)$, and needs no correction, as the expreſſion vaniſhes when $x = o$; writing h for x the expreſſion becomes $2h\,(a + c)$.

Now by the property of the circle

$$a = (\tfrac{1}{4}\, l^2 + h^2) \div h = \frac{241}{4} \text{ and } c = (\tfrac{1}{4}b^2 + h^2) \div h = 40;$$

wherefore $2h\,(a + c) = 8082$, the convex ſurface of the groin.

Again, from what is delivered at Art. 155. Vol. 1. Simpſon's Fluxions,

$$4\dot{x}\sqrt{(ax - x^2)} \times \sqrt{(cx - x^2)} = 4x\dot{x}\sqrt{\left\{ ac - (a + c)x + x^2 \right\}}$$

$=$

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$= \text{the fluxion of the solidity} = \dfrac{4acx\dot{x} - 4(a+c)\,x^2\dot{x} + 4x^3\dot{x}}{\sqrt{\left\{ac - (a+c)\,x + x^2\right\}}}$

by throwing the surd into the denominator.

And the fluent thereof when properly corrected is

$$\left\{\frac{(a+c)^2}{2} - \frac{4ac}{3}\right\}\sqrt{(ac)} + \frac{4x^2}{3}\sqrt{\left\{ac-(a+c)\,x+x^2\right\}}$$
$$- \frac{(a+c)\,x}{3}\sqrt{\left\{ac-(a+c)\,x+x^2\right\}}$$
$$- \left\{(a+c)^2 - \frac{4ac}{3}\right\}\sqrt{\left\{ac-(a+c)\,x+x^2\right\}}$$
$$+ \left\{ac(a+c) - \frac{(a+c)^3}{4}\right\} \times \text{h.l.} \left\{\frac{\frac{a+c}{2} - x - \sqrt{\left\{ac-(a+c)x+x^2\right\}}}{\frac{a+c}{2} - \sqrt{(ac)}}\right.$$

and writing h in the place of x, the expreſſion becomes

$$\left\{\frac{(a+c)^2}{2} - \frac{4ac}{3}\right\}\sqrt{(ac)} + \frac{4h^2}{3}\sqrt{\left\{ac-(a+c)\,h+h^2\right\}}$$
$$- \frac{(a+c)\,h}{3}\sqrt{\left\{ac-(a+c)\,h+h^2\right\}}$$
$$- \left\{(a+c)^2 - \frac{4ac}{3}\right\}\sqrt{\left\{ac-(a+c)\,h+h^2\right\}}$$
$$+ \left\{ac(a+c) - \frac{(a+c)^3}{3}\right\} \times \text{h.l.} \left\{\frac{\frac{a+c}{2} - h - \sqrt{\left\{ac-(a+c)h+h^2\right\}}}{\frac{a+c}{2} - \sqrt{(ac)}}\right.$$

And this is the correct expreſſion for the ſolid content or vacuity of the groin. By a careful calculation, the numerical value of the ſolidity of the groin from the preceding expreſſion is found to be 1483·688422 cubic inches.

XIX. QUESTION 289, *anſwered by Mr.* I. T. M'Doneld.

LEMMA. The locus of a point within a given trapezium, whence perpendiculars demitted on the ſides are together equal to a given length, is a right line given by poſition.

DEMON.

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DEMON. Let ABCD (fig. 648, pl. 32.) be a given trapezium. At the distance DE, or CH = the given sum of 4 perpendiculars draw EF, GH ‖ to AD, BC, meeting the other sides produced in the points I, F, G, K, and bisect the angles at those points by right lines meeting the sides in L, M, N, O; join LM and NO intersecting BC in P and AD in Q; draw PQ and it is the locus. For, through L and M draw RS, TV parallel and equal to DE; LW ⊥ to CD, and MY ⊥ to AB; then because the ∠ K is bisected, LW = LS and LW + LR (= MT + MY) = DE, and LM is the locus of a point, whence perpendiculars demitted on the 3 sides AB, AD, CD are together equal to the sum of the perpendiculars on the 4 sides; which I thus prove: through any point p in LM draw qr ‖ DE, also ps ‖ AB, pn ⊥ to AB and po ⊥ to CD, also Md ‖ EF;
then LS — MV = LW — MY = Ld, and pr — MV = ph.

But by sim fig. LW : MY :: po : Ms;
dividendo LW : Ld :: po : ph (by sim. △s).

Hence po exceeds Ms as much as pr exceeds MV,
and $pn + po$ = MY + ph = pr; therefore $pq + pn + po$ = qr or DE.

In like manner it may be proved that ON is the locus of a point whence perpendiculars demitted on AB, BC, CD, are together equal to CH or DE. Further, through any point Q′ in PQ, draw ad ‖ AD meeting DE in e; then it is too manifest to need any demonstration here, that if $aBCd$ were the given trapezium and eE the given sum of the perpendiculars, that Q′ would be the position of Q, O′N′ of ON, and that perpendiculars from Q′ on the sides aB, BC, Cd, would together be equal to eE, to which add eD or Qf, and DE is the sum of the perpendiculars on the 4 sides demitted from the point Q′; and the same may be proved of any other point in the line PQ. *Q. E. D.*

The locus PQ (fig. 649, pl. 32.) of the required point being determined for the sum of the perpendiculars, by the preceding Lemma; next find two right lines RY, RZ (by Prop. XVI. of Stewart's Gen. Theo.) that will be given by position such that the sum of the squares of perpendiculars demitted from any point on the sides of the trapezium may be equal to 4 times the sum of the squares of perpendiculars from the same point on RY, RZ, together with a given space, viz. the sum of the squares of the perpendiculars Rs, Rt, Rv, Rw (Vide Lowry's Sol. pa. 329, vol. I.) The problem will then be reduced to this: to find a right line given by position within a given angle YRZ, such that the sum of the squares of perpendiculars from that point on RY, RZ may be equal to a given space, viz. ¼ of the difference between the sum of the squares Rs, Rt, Rv, Rw, and the sum of the squares, of perpendiculars from the required point on the sides of the tra-

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pezium which may be done thus: From the point of interſection a, of PQ and RY, demit $ab \perp$ to RZ, produce YR till $Rc = ab$, draw Pd alſo $\perp$ to RZ, and $Pe \parallel ac$ meeting $Re \perp$ to ac in e, and produce it till $ef = Pd$; join cf, and take $Rg =$ ſide of a ſquare equal to the given ſpace; laſtly draw $gO \parallel cY$ meeting PQ in O, and O is the point required to be found.

For if Ok, Oi, be made $\perp$ to RY, RZ,
it is evident from the conſtruction that $Oi = gk$, and $Ok = Rk$, and $Rk^2 + gk^2 = Ok^2 + Oi^2 = Rg^2 =$ given ſpace.

Moreover, by the abovementioned propoſition
$Rs^2 + Rt^2 + Rv^2 + Rw^2 + 4Ok^2 + 4Oi^2 = Os^2 + Ot^2 + Ov^2 + Ow^2 =$ the ſum of the ſquares of the 4 perpendiculars given per queſtion.

SCHOL. The ſum of the ſquares of Rs, Rt, Rv, Rw is a minimum.

XX. QUESTION 290, *anſwered by Mr.* John Dawes.

From the data are found the latitude $= 18° 53' 42''\frac{1}{2}$ and the ſun's declination $= 19° 47' 25''$.

When the azimuth circle ZON (fig. 645, pl. 31.) becomes a tangent to the parallel of declination d, d, that point is the poſition of the ſun when a maximum in azimuth. In the right angled triangle POZ there are given PZ = co-latitude and PO = co-declination to find the angle $OPZ = 17° 58' 4''$; hence the required time is 1ob. $48' 7'' 44'''$.

The ſame, by Captain Geo. Gorry.

It appears from Art. 137, of Mr. Gregory's Aſtronomy, that when the ſun's declination is equal to the difference between the co-latitude and 18°, then the parallel of declination will juſt touch the crepuſculum circle, and the twilight will juſt continue all night. Hence $37° 47' 25''$ (co-latitude of Cambridge) $- 18° = 19° 47' 25''$ north declination of the ſun at the commencement of the period when twilight continues all night, this anſwers pretty well to May 20th 1803. The latitude of the place where the ſun's maximum azimuth from the north is ſought, is $\frac{37° 47' 25''}{2} =$

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$= 18° 53' 42''\frac{1}{2}$. This latitude is therefore less than the sun's declination; and the case corresponds with what is shewn at Art. 135, of the work above referred to; where it appears that the azimuth circle ZO will touch the parallel of the sun's declination in the point O and there will be given in the spherical triangle ZOP, right angled at O, the co-latitude $ZP = 71° 6' 17''\frac{1}{2}$, and the co-declination $OP = 70° 12' 35''$, to find ZPO the hour angle from noon; that is, tang. ZP : tang. OP :: radius : cosine of $P = 17° 58' 4''$; this converted into time gives 1h. 11m. 52s. 16t. time from noon; therefore 10h. 48m. 7s. 44t. is the time in the morning when the azimuth from the north is the greatest, at the given day and place.

The same, by Tyro Philomatheticus.

It is manifest that the beginning of the period mentioned in the question will be when the sun's declination is $= 37° 47' 25''$ (co-lat. of Cambridge) $— 18° = 19° 47' 25''$, and the latitude of the place is given $= 18° 53' 42''\frac{1}{2}$; therefore let HZHN be the meridian, H H the horizon, SP the axis, Æ Æ the equator, draw the parallel of declination *d*, *d*, and by Prob. XLV. Simpson's Problems, at the end of his Geometry, describe an azimuth circle to touch the parallel *dd*, and the point of contact O will evidently give the sun's greatest distance from the north; then an hour circle being drawn through O will shew the time required. Now, because by spherics the radius *Ad* of the parallel *dd*, is a tangent to the primitive HZP at *d*, we have $Ad^2 = AP \times AS$, but $Ad = AO$; therefore $AO^2 = AP \times AS$, whence AO is a tangent to the hour circle POS at O; but by Simpson's Geometry III. 8. Cor. 2. AO produced passes through the centre B of the azimuth circle ZON and consequently is perpendicular to ZO, and therefore ZOP is a right angle; hence we have, as cotang. lat. : cot. declin. :: radius : cosine of the hour from noon $= 17° 58' 3''\frac{1}{2} =$ 1h. 11m. 52s. 14t. the time required from noon.

It was also answered by Mr. Marrat.

XXI. QUESTION 291, *answered by* Quinbus Flestrin.

Let ACD (fig. 647, pl. 81.) be a semicircle, AC, CD, and DB arcs into which its circumference is subdivided. Draw the chords of these arcs and the radii CO, DO. The chords AC,

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CD, and DB have obviously the same ratio as their halves; and these half chords are respectively the sines of the angles $\frac{1}{2}$AOC, $\frac{1}{2}$COD, and $\frac{1}{2}$DOB.

And $\frac{1}{2}$AOC + $\frac{1}{2}$COD + $\frac{1}{2}$DOB = a right angle.

This being premised, put AB = $2r$, AC = x, DB = y; the ratio of AC to CD that of r to b; viz.

AC : CD :: $r : b$ :: $2r$ = AB : $2b$; whence

$2b \times$ AC = $2bx$ = AB × CD. Also let the given ratio of AC to BD be denoted by that of r to c; viz.

AC : BD :: $r : c$, whence $r \times$ BD = $c \times$ AC, or $ry = cx$.

Then by Eu. I. 47. BC = $\sqrt{(4r^2 - x^2)}$; and AD = $\sqrt{(4r^2 - y^2)}$; And by Emerson's Geometry, Book IV. Prop. 23.

BC × AD = AB × CD + AC × CD, which being expressed algebraically from what has been deduced is

$\sqrt{(4r^2 - x^2)} \times \sqrt{(4r^2 - y^2)} = 2bx + xy$,

and by squaring the equation

$$\left.\begin{array}{l}(4r^2 - x^2) \times (4r^2 - y^2) \\ \text{or } 16r^4 - 4r^2(x^2 + y^2) + x^2y^2\end{array}\right\} = 4b^2x^2 + 4bx^2y + x^2y^2;$$

or, $bx^2y + r^2x^2 + r^2y^2 + b^2x^2 = 4r^4$, and exterminating y by means of the equation $ry = cx$ it becomes

$$\frac{bcx^3}{r} + (r^2 + b^2 + c^2) \times x^2 = 4r^4,$$ from whence x may be found.

The same, by Scotieus.

Let φ and ψ denote two of the angles into which the right angle is to be divided; then the sines of the three angles required will be

Cof. $(\varphi + \psi)$, fin. φ, fin. ψ.

So that, putting x, n, p to denote the given ratios, we have

Cof. $(\varphi + \psi) = n$ fin $\varphi = p$ fin ψ.

But cof. $(\varphi + \psi)$ = cof. φ cof. ψ — fin. φ fin. ψ, therefore fubftituting for fin. ψ its value $\frac{n}{p}$ fin. φ, and tranfpofing,

$$\text{cof. } \varphi \text{ cof. } \psi = \frac{n}{p} \text{ fin. } \varphi^2 + n \text{ fin. } \varphi.$$

Hence,

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Hence, squaring and substituting $1 - \text{fin. } \varphi^2$ for $\text{cos. } \varphi^2$, also $1 - \frac{n^2}{p^2} \text{ fin. } \phi^2$ for $\text{cos. } \psi^2$ we have this equation

$$(1 - \text{fin. } \varphi^2)\left(1 - \frac{n^2}{p^2} \text{ fin. } \phi^2\right) = \left(\frac{n}{p} \text{ fin. } \phi^2 + n \text{ fin. } \varphi\right)^2$$

which by proper reduction and putting $\text{fin } \varphi = x$ becomes

$$\frac{2n^2}{p^2} x^2 + \left(n^2 + \frac{n^2}{p^2} + 1\right) x^2 - 1 = 0,$$

Or, putting $y = \frac{1}{x}$.

$$y^2 - \left(n^2 + \frac{n^2}{p^2} + 1\right) y - \frac{2n^2}{p^2} = 0$$

a cubic equation wanting the second term and therefore ready to be resolved by known rules.

Ingenious solutions to this question were also received from Messrs. Cunliffe *and* Whitley.

XXII. QUESTION 292.

NOT ANSWERED.

XXIII. QUESTION 293, *answered by Mr.* Bazley, *Bolton.*

It is well known that the sum of n terms of the series

$$x^1 + x^2 + x^3 + \&c. \ldots\ldots x^{n+1} = \frac{x^{n+2} - x^2}{x - 1};$$

take the second fluxion and it becomes

$$1.2 + 2.3x + 3.4x^2 + 4.5x^3 + \&c. \ldots n(n+1)x^{n-1}$$

$$= \text{the second fluxion of } \frac{x^{n+2} - x^2}{x - 1}$$

$$= \left\{ \begin{array}{l} \dfrac{(n+1)(n+2)x^n - 2}{x-1} + \dfrac{2(x^{n+2}-x^2)}{(x-1)^3} \\ -\dfrac{2\left\{(n+2)x^{n+1}-nx\right\}}{(x-1)^2}; \end{array} \right.$$

therefore the sum of n terms of the propoſed ſeries is

$$= x \left\{ \begin{array}{l} \dfrac{(n+1)(n+2)x^n - 2}{x-1} - \dfrac{2\left\{(n+2)x^{n+1}-nx\right\}}{(x-1)^2} \\ + \dfrac{2(x^{n+2}-x^2)}{(x-1)^3}. \end{array} \right.$$

GENERALLY. Let $x^m + x^{m+1} + x^{m+2} + \&c.$ $x^{m+n-1} = s$,

then $1.2.3\ \&c....m+2.3.4\ \&c....(m+1)x + 3.4.5\ \&c....(m+2)x^2 + 4.5.6\&c...(m+3)x^3 + \&c...n(n+1)(n+2)\&c...(m+n-1)x^{n-1}$ is equal to the mth fluxion of s; ſuppoſing $x = 1$.

The ſame, anſwered by Mr. Cunliffe.

Firſt $x^2 + x^3 + x^4 + x^5 + \&c. \ldots = \dfrac{x^2}{1-x}$;

and $x^{n+2} + x^{n+3} + x^{n+4} + x^{n+5} + \&c. \ldots = \dfrac{x^{n+2}}{1-x}$;

The latter ſeries is evidently a continuation of the former after n leading terms; wherefore their difference will be

$x^2 + x^3 + x^4 + x^5 + \&c.$ to n terms $= \dfrac{x^2 - x^{n+2}}{1-x}$;

the fluxion of this expreſſion divided by $\dot{x}$ is

$2x + 3x^2 + 4x^3 + 5x^4 + \&c.$ to n terms

$= \dfrac{(n+1)x^{n+2} - (n+2)x^{n+1} + 2x - x^2}{(1-x)^2} = \dfrac{v}{(1-x)^2}$, by putting

$v =$

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$v = (n+1)x^{n+2} - (n+2)\,x^{n+1} + 2x - x^2$.

Again taking the fluxion of the preceeding expreſſion and dividing by $\frac{\dot{x}}{x}$ there will be had

$1.2\dot{x} + 2.3x\dot{x} + 3.4x^2\dot{x} + 4.5x^3\dot{x}$ + &c. to n terms

$$= \frac{\frac{x\dot{v}}{\dot{x}} \times (1-x) + 2xv}{(1-x)^2} = \frac{x\dot{v}}{\dot{x}(1-x)^2} + \frac{2xv}{(1-x)^2}.$$

Now by the ſubſtitution

$v = (n+1)\,x^{n+2} - (n+2)\,x^{n+1} + 2x - x^2$, taking the fluxion of this equation and dividing by $\frac{\dot{x}}{x}$ gives

$$\frac{x\dot{v}}{\dot{x}} = (n+1)(n+2)x^{n+2} - (n+1)(n+2)x^{n+1} + 2x - 2x^2;$$

wherefore $1.2\dot{x} + 2.3x\dot{x} + 3.4x^2\dot{x} + 4.5x^3\dot{x}$ + &c. to n terms

$$= \frac{x\dot{v}}{\dot{x}(1-x)^2} + \frac{2xv}{(1-x)^2}$$

$$= \left\{ \begin{array}{l} \dfrac{(n+1)(n+2)x^{n+2} - (n+1)(n+2)x^{n+1} + 2x - 2x^2}{(1-x)^2} \\ + \dfrac{2(n+1)x^{n+3} - 2(n+2)x^{n+2} + 4x^2 - 2x^3}{(1-x)^2} \end{array} \right.$$

$$= \left\{ \begin{array}{l} \dfrac{2x - (n+1)(n+2)x^{n+1}}{(1-x)} \\ + \dfrac{2(n+1)x^{n+3} - 2(n+2)x^{n+2} + 4x^2 - 2x^3}{(1-x)^2} \end{array} \right.$$

Which is a correct expreſſion for the ſum of n leading terms of the propoſed ſeries.

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The same, unfolved by Tyro Philomathematicus.

This question is a particular case of Prob. XXXI. page 286, Emerson's method of Increments, wherein it is required to find S the sum of n terms of the series
$Ma + Na^2 + Oa^3 + Pa^4$ &c.....Putting v for the coefficient of a in the $(n+1)$th term he gets

$$S = \frac{a^{n+1}v}{a-1} - \frac{a^{n+2}\dot{v}}{(a-1)^2} + \frac{a^{n+3}\ddot{v}}{(a-1)^3} - \frac{a^{n+4}\dddot{v}}{(a-1)^4} \text{ &c.}$$

$\dot{v}$, $\ddot{v}$, and $\dddot{v}$ being the 1st, 2nd and 3d increments of v.

In the example before us $a = x$, $v = (n+1)(n+2)$, $\dot{v} = 2(n+2)$, $\ddot{v} = 2$, and $\dddot{v} = 0$; therefore the sum of n terms is

$$S = \frac{(n+1)(n+2)x^{n+1}}{x-1} - \frac{2(n+2)x^{n+2}}{(x-1)^2} + \frac{2x^{n+3}}{(x-1)^3};$$

but this requires correction, for when $n = 0$,

then $S = \frac{2x}{x-1} - \frac{4x^2}{(x-1)^2} + \frac{2x^3}{(x-1)^3}$;

whence the sum corrected is $\frac{(n+1)(n+2)x^{n+1}}{x-1}$

$$- \frac{2(n+2)x^{n+2}}{(x-1)^2} + \frac{2x^{n+3}}{(x-1)^3} - \frac{2x}{x-1} + \frac{4x^2}{(x-1)^2} - \frac{2x^3}{(x-1)^3},$$

this reduced to a common denominator, gives

$$S = \frac{n(n+1)x^{n+3} - 2n(n+2)x^{n+2} + (n+1)(n+2)x^{n+1} - 2x}{(x-1)^3}$$

If x be a fraction the theorem will be better expressed with the signs changed.

This Question was ingeniously answered by Messrs. Limenus, Marrat, Quinbus Flestrin, *and* Thornoby.

XXIV.

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XXIV. QUESTION 294, *answered by Mr.* Cunliffe.

ANALYSIS. Let ABCD (fig. 651, pl. 32.) be the trapezium, its diagonals AC, BD interfecting each other in I, and fuppofe the latter divides the former in the given ratio. Draw I*m*, I*p*, refpectively parallel to BC, AB, and terminating in thefe lines in *m* and *p*. Alfo draw I*n*, I*q*, refpectively parallel to DC, AD, and terminating in thofe lines in *n* and *q*. Then it is obvious that the fide AB is divided as *m* in the given ratio of the parts of the diagonal AI, IC, and therefore B*m* is given, and for the fame reafon, B*p* is given, or known, as likewife D*n* and D*q*.

Then by prop. 36, Lawfon's Theorems

$$BI^2 + AI \times IC = BA \times Bm + BC \times Bp$$
$$= \text{a given magnitude; and}$$
$$DI^2 + AI \times IC = DA \times Dn + DC \times Dq$$
$$= \text{a given magnitude.}$$

And the difference of thefe will give

$DI^2 - BI^2 =$ a given magnitude. Now with the centres *q* and C, and the radii *q*I, CB let two circles be defcribed; then the queftion is evidently reduced to what follows; viz. From the given point D to draw a right line to cut the circles whofe radii are *q*I, and CB in I and B fo that $DI^2 - BI^2$ may be of a given magnitude.

Quinbus Flefirin *alfo anfwered this queftion.*

XXV. QUESTION 295, *anfwered by* Scoticus.

Fig. 675. Pl. 33. Suppofe a line drawn through the given point P as required. Through the fame point draw any other line given by pofition, cutting the given lines AL, BM, CN, DO, &c. in *a*, *b*, *c*, *d*, &c. Then the lines P*a*, P*b*, P*c*, P*d*, &c. will be given in magnitude. Join PB, PC, PD, &c. cutting A*a*, one of the given lines in β, γ, δ, &c. The lines PB, PC, PD, &c. will be given by pofition, and therefore the points β, γ, δ, &c. will alfo be given.

From fimilar triangles, P*a* : P*b* :: Pβ : PB :: βL : BM,

therefore $BM = \frac{Pb}{Pa} \cdot \beta L$, and $BM^2 = \frac{Pb^2}{Pa^2} \cdot \beta L^2$.

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In like manner $CN^2 = \frac{Pc^2}{Pa^2}.\gamma L^2$, alſo $DO^2 = \frac{Pd^2}{Pa^2}.\delta L^2$, &c.

Now by hypotheſis

$AL^2 + BM^2 + CN^2 + DO^2 + \&c. =$ a given ſpace,

therefore alſo

$$AL^2 + \frac{Pb^2}{Pa^2}.\beta L^2 + \frac{Pc^2}{Pa^2}.\gamma L^2 + \frac{Pd^2}{Pa^2}.\delta L^2 + \&c. = \text{a given ſpace.}$$

But it appears from Dr. Stewart's XV General Theorem, (See Math. Rep. Vol. I. Pa. 349.) that a point X may be found in Aa, ſuch, that wherever L be taken

$$\left.\begin{aligned} AL^2 + \frac{Pb^2}{Pa^2}.\beta L^2 + \frac{Pc^2}{Pa^2}.\gamma L^2 \\ + \frac{Pd^2}{Pa^2}.\delta L^2 + \&c. \end{aligned}\right\} = \left\{\begin{aligned} \frac{Pa^2 + Pb^2 + Pc^2 + Pd^2 + \&c.}{Pa^2}.XL^2 \\ + \text{a given ſpace:} \end{aligned}\right.$$

Hence it is evident that

$$\frac{Pa^2 + Pb^2 + Pc^2 + Pd^2 + \&c.}{Pa^2}.XL^2 = \text{a given ſpace.}$$

Therefore XL^2 and conſequently XL is given. Thus the conſtruction of the problem is reduced to the determination of the point X, which may be found as in the place already quoted.

XXVI. QUESTION 296, *anſwered by Mr.* Cunliffe.

Put $m^2 - n^2 = x$, and $2mn = y$, then

$x^2 + y^2 = (m^2 + n^2)^2$ which is a rational ſquare.

It therefore only remains to find ſuch a relation of m and n as ſhall make

$$x^2 + xy + y^2 = (m^2 + n^2)^2 + 2mn \times (m^2 - n^2)$$
$$= m^4 + 2m^3n + 2m^2n^2 - 2mn^3 + n^4 = \text{a ſquare:}$$

Aſſume $m^2 + mn + n^2$ for its root, then will

$$m^4 + 2m^3n + 2m^2n^2 - 2mn^3 + n^4 = (m^2 + mn + n^2)^2$$
$$= m^4 + 2m^3n + 3m^2n^2 + 2mn^3 + n^4;$$

Whence $m^2n^2 = -4mn^3$, and $m = -4n$.

But this relation of m and n would give the value of y negative; therefore in order to obtain poſitive values for both x and y, put $m = v - 4n$, then will

$$x^2 + xy + y^2 = (m^2 + n^2)^2 + 2mn \times (m^2 - n^2)$$
$$= v^4 - 14v^3n + 74v^2n^2 - 178vn^3 + 169n^4 = \text{ſq.}$$

Aſſume

Assume $(v^2 - 7vn - 13n^2)$ for the root of the last expression, that is, put

$$v^4 - 14v^3n + 74v^2n^2 - 178vn^3 + 169n^4 = (v^2 - 7vn - 13n^2)^2$$
$$= v^4 - 14v^3n + 23v^2n^2 + 182vn^3 + 169n^4;$$

whence $51v^2n^2 = 360vn^3$, or $v = \frac{120n}{17}$;

put $n = 17$, then $v = 120$, and $m = v - 4n = 52$;
whence $x = m^2 - n^2 = (52)^2 - (17)^2 = 2415$
and $y = 2mn = 52 \times 34 = 1768$.

A solution to the following question may be easily effected from what has been done, viz. To find two fractions of such a kind that both their sum, and sum of their squares may be a rational square; also if either of them be added to the square of the other, the sum shall be a rational square; and furthermore if their product be subtracted from either of them, the remainder shall be a rational square.

Let the two fractions be denoted by $\frac{x}{x+y}$ and $\frac{y}{x+y}$; for their sum will obviously be 1, which is a square number, and the sum of their squares will be $\frac{x^2+y^2}{(x+y)^2}$, which is to be a square, therefore $x^2 + y^2$ must be a square, and if either of them be added to the square of the other the sum will be $\frac{x^2+xy+y^2}{(x+y)^2}$ which is to be a square, therefore $x^2 + xy + y^2$ must be a square. The two remaining conditions will be answered by the assumption or notation, viz. if their product be taken from either of them, the remainder shall be a square. The question is therefore reduced to finding such values of x and y as shall make $x^2 + y^2$, and $x^2 + xy + y^2$ both rational squares, which, from what is before done, will be when $x = 2415$, and $y = 1768$, and the required fractions will be $\frac{x}{x+y} = \frac{2415}{4183}$ and $\frac{y}{x+y} = \frac{1768}{4183}$.

Neat solutions were received from Limenus *and* Whitley.

XXVII. QUESTION 297, *answered by Mr.* J. H. Swale.

Fig 653. Pl. 32. CONSTR. Join PQ and make the $\triangle$ PQR of the given magnitude; through R draw the indefinite right line RG

RG parallel to PQ. Then through Q, by Prob. 37, Simpson's Geometry, draw the right line QGBC cutting the lines RG, AB, and AC in G, B, and C, so that BC may be equal to QG, and the thing is done.

DEMONS. Draw the lines FG, PB, and PC. Then BC = QG by construction; therefore the △ BPC = △ QPG = △ QRP, the given magnitude. *Q. E. D.*

And thus nearly are the answers given by Messrs. Cunliffe, M'Donald, *and* W. Smith.

XXVIII. QUESTION 298, *answered by Mr.* J. T. M'Donald.

CONSTRUCTION. Fig. 684. Pl. 33. Bisect AP in H, and take HI such that rect. PI·IH = rect. BP·PH; draw ID ‖ to EC meeting AC in D, through which draw PDE.

DEMONSTRATION. Bisect the base AB by CG, and ‖ thereto draw CK; also draw EQ and DF ‖ to AB, and join HL which passes through D. Then because DEF is a maximum, its half DML is also a maximum. If not, through any other point *d* in AC draw H*de*, and *fdg* ‖ and equal to MD, meeting KD produced in *g*; bisect KG in N, and draw N*q* ‖ CG, meeting DM in O, *fg* in *o*, DL in *p*, and *ke* in *q*, also draw *ghi* ‖ and equal to DL, and *kr* ‖ and equal to D*g* or O*o*, and *dmn* ‖ and equal also to D*g* meeting DL in *m*. Now by Construction

PI : PB :: HP or HA : HI; dividendo,
PI : DF :: HA : AI, or
PI : DM :: HA : AK, and
HK : KN :: HA : AK; convertendo,
HK : HN :: HA : HK = QL : DM = CL : CM, therefore
HK : HN = CL : CM, and consequently
D*k* : *pq* = *dm* : *dn* or O*o*;
but *dm* is greater than D*k*, and therefore O*q* or *kr* exceeds *pq*.

Then (Euc. I. 35), the parallelograms DM*fg*, DL*eg* are equal, from which if the common triangle D*fe* be taken away and L*pf* added, we have the △ DML equal to *ghi* which exceeds the △ *dfe* as much as *ihg* is less than *ghk*. Further draw any other line H*vy* below D meeting N*q* in *x* and DK in *a*, and draw *vcw* ‖ DM meeting N*q* in *c* and DK in *b*; then

D*a* : *px* :: HK : HN :: HA : HK :: CL : CM,

which, because ∠ C*vy* is greater than CDL is a less ratio than that of C*y* : C*w* or D*a* : O*c*; therefore *px* is greater than O*c*, and the

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the ſpace LD*oy* than MD*ow*, and DML than *way*. Hence the triangle DML and conſequently DEF is a maximum.

Q. E. D.

The ſame, by Mr. Wm. Smith, *the Propoſer.*

LEMMA. 1ſt. Let ABC (fig. 657, pl. 32.) be any △, P a given point in BA produced, draw any line PDE to cut AC, BC in D and E reſpectively, and draw EF ‖ AB meeting AC in F. Then if AQ be taken in the right line CA equal to DF and PQ be joined, meeting EL ‖ AC in L; the locus of all the points L will be a parabola whoſe diameters are ‖ to AC, and the parameter (namely of the diameter biſecting BA) of which is to CH (PH being ‖ BC and meeting CA in H) as (joining PC and drawing EW ‖ AC to meet it in W) FE² is to FC·EW.

DEMONSTRATION. By the ſimilar △s PDH, EDC and PAD, EDF the rect. HD·DF = rect. AD·DC therefore (producing EL to meet PH in Y and joining PF meeting EL in O) by parallels rect. YE·EO = KE·EW, that is, ſince AQ = DF, the rect. YE·KL = rect. KE·EW. Therefore YE·LK : BK·KA :: KE·EW : BK·KA :: FC·EW : FE²; but per Emerſon's Conics, Prop. 25th, Bk. IIId. (P being the parameter, &c.) P·LK = BK·KA; therefore P : YE :: FE² : FC·EW. Therefore the locus of L is a parabola, whoſe parameter is to HC (YE) :: FE² : FC·EW. *Q. E. D.*

LEMMA 2d. Let P be a given point in the ſide SA produced of any given △ SAR. Then I ſay, if PQL be drawn meeting AR, RS in Q and L reſpectively, and AL be joined, the △ AQL will be a maximum, when (drawing LG ‖ AS meeting AR in G) AQ = RG.

DEMONSTRATION. Draw PZ ‖ AR meeting SR in Z, PV ‖ SR meeting RA in V, and ZX ‖ PA meeting AR in X. Then by ſimilar △s PLZ, PQV, rect. ZL·QV = rect. PZ·PV, and therefore the rect. XG·QV = rect. AV·PZ, that is, wherefoever PQL be drawn the rect. XG·VQ is given. Again the △ ALQ = ½AP·GQ (LK — QA), therefore, when the △ AQL is a max. the line GQ is a max. or the ſum of XG and VQ a *minimum*, but the rect. XG·VQ is given, therefore XG + VQ is a *minimum*, when XG = VQ, and when XG = VQ, therefore the line GQ and the △ AQL are both the greateſt poſſible, but by ‖s XR = AV, therefore RG = AQ. *Q. E. D.*

Hence the problem may be thus conſtructed:

Draw any line *fe* ‖ AB and *ew* ‖ AC meeting PC in *w*, determine the line C*d*, ſuch that twice C*d* may be to CH :: *fe*² : *f*C·

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cw. Biſect AB in M, and produce PM to K, ſo that drawing KE || AC, the rect. FK KM may be equal to $Cd \cdot KE$ (the method of doing which is ſo ſimple that it need not be detailed); join PE meeting AC in D, and draw EF || AB, ſo ſhall DEF be the maximum △ required.

DEMONSTRATION. On AB deſcribe a parabola, as in Lemma 1ſt, the parameter of whoſe diameter to the ordinate AB may be equal to twice CL, and its diameter || AC. Produce EK to meet PC, and the parabola in W and L reſpectively, draw the tangent SLR meeting AB, AH in S and R reſpectively, draw alſo LG || AB; join PL (meeting AB in Q) and AL, and through M the middle of AB draw INM *aqgxy* meeting EF, ED, AB, AL, PL, LG the parabola and tangent SR in the points I, N, M, *a*, *q*, *g*, *x*, and *y* reſpectively. Then by conſtruction, rect. PK. KM = $KE \cdot Cd$: but PK is to KE as FE : FD. Therefore rect. $FE \cdot EI = FD \cdot Cd$, that is, by ||s, Lemma 1ſt, and conſtruction rect. $GL \cdot Lg = AQ \cdot Cd$; and therefore ſince GL : AQ :: $gL \cdot aq : gL^2 = aq \cdot Cd$, that is, $gL^2 = aq \cdot \frac{1}{2}P$ (Parameter) but $gL^2 = gx$. P, therefore, *aq* (= twice *gx*) = (per nature of the tangent to the parabola) *yg*, and thence AQ = RG. Therefore, per Lemma 2d, the △ AQL is a maximum within the tangential triangle ARS, and conſequently is a max. in the inſcribed parabola or curve: but ſince AQ = DF the △ AQL is always = to △ EDF; therefore the △ EDF is a maximum alſo. *Q. E. D.*

The ſame by Mr. John Wright, Jun. *of Norley.*

ANALYSIS. Suppoſe the thing done, and perpendicular to CB (fig. 658, pl. 32.) draw PK meeting AK drawn parallel to BC in K and BC in I; join CK, and through D, parallel to BC, let a right line be drawn, cutting AB in L, PK in G, and CK in S; demit SM perpendicular to BC, alſo draw FS cutting BC in V. Then by reaſon of parallel lines,

GK : KI :: LD : EC, and GK : KI :: GS : IC,

and therefore, by equality of ratios,

GS : IC :: LD : EC, or GS : LD :: IC : EC:

alſo, becauſe of the ſim. △s VMS, PGS; EDF, PLD, it will be

VM : GS :: SM (GI) : PG :: EF : LD; wherefore

VM : EF :: GS : LD :: IC : BC *(ſupra)* a given ratio, conſequently the ratio of VM : EF is given; and becauſe LS is parallel to BC, the triangle VSM has to the triangle EDF the ſame given ratio of VM to EF, therefore, when the triangle EDF is a maximum, the right angled triangle VSM will be a maximum alſo.

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On PK describe the semicircle PYQK, cutting SL and CB produced in Q and Y; draw QI, also PQ cutting IY in R; and complete the parallelogram MSQH. Then, by similar triangles,
$\triangle$ PGS : $\triangle$ VSM :: PG^2 : SM^2 (QH^2)
:: QG^2 ($PG.GK$) : HR^2; that is,
HR^2 : $\triangle$ VSM :: $PG.GK$: $\frac{1}{2}PG.GS$ :: GK : $\frac{1}{2}GS$ a given ratio, and therefore when the triangle VSM is a maximum, HR^2 will be a maximum, and consequently HR; but when this is a maximum, the triangle RQI must be so; as is evident from the consideration of the similar triangles PIR and HQR, the side PI being constant. Now, if the tangent TN, terminating in IK and IY produced, be drawn to the circle at Q, it will readily appear, from either of the solutions of Question 595, Vol. III. of Hutton's Diarian Miscellany, that when TH is equal to RI, the triangle RQI is greater than any other which can be formed within the triangle TNI, by lines drawn from P and I, and it must evidently then be greater than any other which can be so formed within the semi-segment IKQY. Having therefore shewn that the triangle RIQ, and of consequence the required one EDF, is a maximum when TH = RI, let QK be joined, and, O being the centre of the semi-circle, draw the radius OQ; then the triangle QGK is manifestly similar to QRH,
and hence $QG \cdot RH = GK \cdot QH$;
the triangle OGQ is also similar to THQ,
and $OG \cdot QH = QG \cdot TH$ ($QG \cdot RI$) $= QG^2 - QG \cdot RH = PG \cdot GK - KG \cdot QH$;
therefore $PG \cdot GK = QH \cdot (OG + GK) = QH \cdot OK = PG \cdot OK - PI \cdot OK$;
or $PG \cdot OG = PI \cdot OK$ = a given magnitude. Whence the following

CONSTRUCTION. Having drawn PN perpendicular to BC in I, to meet AK parallel to BC in K, bisect PK in O, from which point on OK set of OG such, that the rectangle under OG and PG may be equal to the given one $PI \cdot PO$; from G, parallel to BC, draw GD meeting AC in D; then, if DF parallel to AB meet BC in F, and PD be drawn cutting BC in E, the triangle EDF will evidently be the maximum required.

XXIX. QUESTION 299, *answered by Mr.* Wallace, *Royal Military College.*

ANALYSIS. Suppose the lines drawn as required. From B, (fig. 673, pl. 32.) one of the given points in the circumference

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of the given circle, draw BY to one of the given points in the lines given by position; let BY cut the circle again in C; join AC, and take AD = BY; the points D, C, are evidently both given in position; through these points and the given point Y describe a circle cutting the given line YH in F, join FD cutting AE in K, the point F and therefore the line FK are also given in position. The triangles BHY, AKD, are in all respects equal, for FDCY being a quadrilateral in a circle the outward angle HYB is equal to the inward and opposite angle FDC, or to ADK which is vertical to FDC; the angles CBE, CAE, that is YBH, DAK are also equal, for they are in the same segment of the given circle; therefore the triangles YBH, DAK are equiangular, but they have the sides BY, AK (which are opposite to equal angles in each) equal by construction, therefore the remaining sides of the one triangle are equal to those of the other, so that DK is equal to YH: hence the sum or difference of the squares of XG, YH is equal to the sum or difference of the squares of XG, DK; thus the proposed question is reduced to the following. Let DK, XG be two straight lines given by position and D, X given points in those lines, also let A be a given point without them: It is required to draw a line through A to cut them in K and G so that $DK^2 + XG^2$ may be a given space; or so that $DK^2 - XG^2$ may be equal to a given space. We proceed to consider the question under its new form.

Draw AL parallel to GX cutting KD in L, and draw AM parallel to KD, cutting GX in M, the lines AL, AM are evidently given in magnitude. The triangles KLA, AMG are similar, therefore KL : LA :: AM : MG, hence KL × MG = AL × AM = a given space. Draw GO perpendicular GX, and equal to KL, then the given rectangle KL × GM becomes GO × GM, and since the point M, also the line MG are both given by position, the Locus of O is a given right angled hyperbola having MG for one of its asymptotes; the other will pass through M at right angles to MG. Draw XP perpendicular to XG and equal to DL, thus P will be a given point; draw PQ parallel to XG, meeting OG in Q, then the line PQ is given by position; and since by construction OG = KL, also GQ = PX = DL, we have OQ = DK, now PQ = GX, therefore $DK^2 \pm XG^2 = OQ^2 \pm QP^2$.

Now if $YH^2 + GX^2 = DK^2 + XG^2$ be given, then $OQ^2 + QP^2 = OP^2$ is given, therefore the point O is in the circumference of a circle whose centre is a given point P, and radius a given line PO; but the same point is also in a given hyperbola as has been already shewn, therefore the point O is determined, and hence

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hence the position of the line OG and the line GAE, as was required. If again $YH^2 - GX^2 = DK^2 - GX^2$ be given, then $QO^2 - QP^2$ is given so that in this case the LOCUS of O is another right angled hyperbola whose centre is P, and conjugate axis in PQ, and therefore the problem may be resolved by the interfection of two hyperbolas.

In general, let the relation between XG and YH be expreſſed by any equation whatever involving theſe indeterminate quantities and known magnitudes only, the ſame equation will expreſs the nature of a curve which is the locus of the point O. Thus we have ſhewn how the general problem may be reſolved, not only in the two caſes mentioned in the queſtion, but in innumerable others which might be propoſed. If for example we had XG + YH = a given line then alſo PQ + QO = a given line, or ſince PX, XM are given, MG + GO = a given line, but here we may diſpenſe with the hyperbola, for we have only to determine two lines MG, GO whoſe ſum and rectangle are both given; and ſimilar ſimplifications will apply to other caſes. As to the conſtruction it follows ſo obviouſly from the analyſis that for the ſake of brevity it may be omitted.

XXX. OR PRIZE QUESTION 300, *anſwered by* Limenus, *Bruton, Somerſet.*

LEMMA. If the ſide AB (fig. 659, pl. 32.) of any triangle be divided in P in the duplicate ratio of AC: BC the other two ſides, and perpendiculars are let fall from P on theſe ſides, the ſum of the ſquares of thoſe perpendiculars, $PM^2 + PN^2$, is the leaſt poſſible.

Triangles of the ſame altitude being as their baſes

$\triangle APC : \triangle PCB :: AC \cdot PM : BC \cdot PN :: AP : PB$
$:: \text{(by conſtr.)}\ AC^2 : BC^2$; whence

$PM : PN :: AC : BC$, and $\frac{PM^2}{AP} = \frac{PN^2}{BP}$

Let P be any point between P and B, and ſince

$$AB + \frac{P\rho^2}{AP} + \frac{P\rho^2}{BP} = \left.\begin{matrix} AP + 2P\rho + \frac{P\rho^2}{AP} \\ +BP - 2P\rho + \frac{P\rho^2}{BP} \end{matrix}\right\} = \frac{A\rho^2}{AP} + \frac{B\rho^2}{BP},$$

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AB is leſs than $\frac{Ap^2}{AP} + \frac{Bp^2}{BP}$;

but becauſe of the ſimilar triangles

APM, Apm, and BPN, Bpn,

$\frac{Ap^2}{AP^2} = \frac{pm^2}{PM^2}$, and $\frac{Bp^2}{BP^2} = \frac{pn^2}{PN^2}$, therefore

$AB = AP + BP$ is alſo leſs than $AP \times \frac{pm^2}{PM^2} + BP \times \frac{pn^2}{PN^2}$,

and multiplying the whole by $\frac{PM^2}{AP}$ and its equal $\frac{PN^2}{BP}$,

$PM^2 + PN^2$ is leſs than $pm^2 + pn^2$.

In like manner the ſame thing may be proved when p is between A and P.

Fig. 66o, pl. 32. Draw ab ‖ to the baſe AB at any diſtance therefrom, which divide in R in the duplicate ratio of the ſides, and join CR meeting ab in Q; then ſince

$aQ : bQ$ :: (AR : BR :: $AC^2 : BC^2$ ::) $Ca^2 : Cb^2$, the ſum of the ſquares of QM, QN will be leſs than of thoſe drawn from any other point in ab, by the preceding lemma.

Now divide either of the ſides AC in S in the duplicate ratio of the remaining ſides of the triangle, and join BS meeting CR in Q, and join AQ to meet BC in T; then, becauſe (by Simpſon's Geom. IV. 23.) BT : TC is in the compound ratio of BR : RA, and AS : SC, it is likewiſe in the duplicate ratio of BA : AC. Therefore the ſums of the ſquares of the perpendiculars QL, QM, and QL, QN are reſpectively leſs than thoſe from any other point in lines drawn through Q ‖ to BC, AC. Whence the point Q being ſo determined, that the ſum of the ſquares of every two of the three perpendiculars QL, QM, QN drawn from thence to three ſides is the leaſt, while the third remains conſtant, we may aſſume Q from whence the ſquares of thoſe perpendiculars make together the leaſt ſum.

Moreover the △ ABC being equal to the three AQB, BQC, AQC, its double is $= QL \cdot AB + QM \cdot AC + QN \cdot BC$.

But QM : QN :: (by the demon. to the lem.) $Ca : Cb$ :: CA : CB,

therefore the perpendiculars from Q are as the ſides on which they ſtand, and

QM·AB

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$QM \cdot AB = QL \cdot AC$, and $QN \cdot AB = QL \cdot BC$, whence
$AB \times 2\triangle ABC = QL \cdot AB^2 + QM \times AB \times AC + QN \cdot AB \cdot BC$
$= QL \times (AB^2 + AC^2 + BC^2)$, and
$QL \times 2\triangle ABC = AB \cdot QL^2 + AC \times QL \times QM + BC \cdot QL \cdot QN$
$= AB \times (QL^2 + QM^2 + QN^2)$; wherefore
$AB : QL :: AB^2 + BC^2 + AC^2 : 2\triangle ABC :: 2\triangle ABC : QL^2 + QM^2 + QN^2$.

Q. E. D.

The ſame, by Mr. I. T. M'Doneld.

LEMMA. Let there be any number of known quantities AB, AC, AD, &c. (fig. 682, pl. 33.) and as many variable quantities Ax, Ay, Az, &c. ſuch that the ſum of their rectangles, $AB \cdot Ax + AC \cdot Ay + AD \cdot Az$, ſhall be equal to a given ſpace; then, I ſay, the ſum of the ſquares of the unknown quantities, $Ax^2 + Ay^2 + Az^2$, &c. will be a minimum, when the reſpective factors have one common ratio to each other,
viz. $AB : Ax = AC : Ay = AD : Az$, &c.

Suppoſe $AD \cdot Az$ to be a conſtant magnitude; and, if the ſum of the ſquares of Ax and Ay be not a min. take Ax' leſs than Ax, and Ay' greater than Ay in an inverſe ratio,
that is, $Ax : Ay :: y'y : xx'$, or rect. $Ax \cdot xx' = Ay \cdot y'y$.

Now $Ax^2 - Ax'^2 = Ax \cdot x' + Ax' \cdot x$, and $Ay'^2 - Ay^2 = Ay'y + Ayy'$; the diff. of theſe two quantities is theref. $Ax'x \sim Ay'y$ but rect. $Ay'y$ exceeds rect. $Ax'x$ in as much as the ratio of Ay' to Ax' exceeds that of $x'x$ to $y'y$; and theref. Ay'^2 exceeds Ay^2 more than Ax^2 exceeds Ax'^2; and vice verſâ if Ax' be taken greater than Ax, its ſquare would exceed that of Ax more than Ay^2 would exceed Ay'^2. Hence $Ax^2 + Ay^2$ is the leaſt poſſible when $Ax : AB = Ay : AC$. It is eaſy to conceive that the propoſition is equally true in *more* quantities than two.

This premiſed, the point Q may be determined as follows: find P (fig. 683, pl. 33.) the centre of gravity of the $\triangle$ ABC, whence demit Pm, Pn, Po, perp. to the ſides of the $\triangle$, and find O the centre of a circle paſſing through the points m, n, o, through which draw PQ, making $OQ = OP$ and Q is the point required, as will appear in the following demonſtration of the theorem.

DEMON. Demit the perps. Qp, Qq, Qr, and on the centre O with rad. OP or OQ deſcribe a circle interſecting the perp. from P

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P in a, b, and c. Then, the points m, n, o being equi-diſtant from O, the rectangles amP, Pnb, ocP, and conſequently their equals Qp.Pm, Qq.Pn, Qr.Po are equal to each other; and as Pm, Pn, Po are inverſely as the ſides on which they ſtand, Pm, Pn, Po will be directly as the ſides, (according to the Lemma),

for Qp : Qq = Pn : Pm = AB : BC;

and Qq : Qr = Po : Pn = BC : AC:

Qp^2 + Qq^2 + Qr^2 is therefore a minimum.

Again rectangle Qp.Pm = Qp^2 — rectangle pQd;

rectangle Qq.Pn = Qq^2 + rectangle nbP,

and rectangle Qr.Po = Qr^2 — rectangle ocP;

but as the $\triangle$s AQB and CQA exceed APB and CPA as much as CQB is leſs than CPB, ſo rectangles pQd + ocP = nbP, and therefore Qp.Pm + Qq.Pn + Qr.Po = Qp^2 + Qq^2 + Qr^2.

Now (AB^2 + BC^2 + AC^2) × (Qp.Pm + Qq.Pn + Qr.Po) and and (AB·Pm + BC·Pn + AC·Po) × (AB·Qp + BC·Qq + AC·Po) when actually multiplied, will by proper ſubſtitution of equal rectangles, be found equal to each other. Hence AB^2 + BC^2 + AC^2 : twice the area :: twice the area : Qp^2 + Qq^2 + Qr^2.

Q. E. D.

The ſame, by Tyro Philomatheticus.

Lemma. If in any right line AB, (fig. 661, pl. 32.) a point C be taken, ſuch, that m.AC^2 + n.CB^2 is the leaſt poſſible, then AC is to CB as n to m; m and n being given or conſtant.

Demonstration. For let D be any point in AB, then m.AD^2 + n.DB^2 = m.(AC + CD)^2 + n.(CB — CD)^2 = m.(AC^2 + 2AC·CD + CD^2) + n.(CB^2 — 2CB·CD + CD^2),

or, ſince AC : CB :: n : m, CB = $\frac{m}{n}$. AC,

m.AD^2 + n.DB^2 = m.(AC^2 + 2AC·CD + CD^2) + n·($\frac{m^2}{n^2}$·AC^2 — $\frac{m}{n}$·2AC·CD + CD^2), which is manifeſtly greater than m.AC^2 + n.CB^2 (= m.AC^2 + n.$\frac{m^2}{n^2}$·AC^2) by $\overline{m+n}$·CD^2.

Let

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Let ABC (fig. 662, pl. 32.) be the triangle. and QD, QE, QF the perpendiculars mentioned in the queſtion, and CI perpendicular to AB, and ſuppoſe the point Q to be ſomewhere in the line GQH which is drawn parallel to AB; then ſince QD (let it be what it will) is the ſame in every point of GQH, we now have to find QF and QE, ſuch, that $QF^2 + QE^2$ ſhall be a minimum; the line GQH being parallel to AB, the angle QGF = angle BAC, and the angle QHE = angle ABC, and QFG and QFH are right angles, therefore by ſimilar triangles

$AC : CI :: QG : QF = \frac{CI}{AC} \cdot QG$, and

$BC : CI :: QH : QE = \frac{CI}{BC} \cdot QH$, whence

$\frac{CI^2}{AC^2} \cdot QG^2 + \frac{CI^2}{BC^2} \cdot QH^2 = QF^2 + QE^2 =$ a minimum, and

conſequently by the Lemma ($\frac{CI^2}{AC^2}$ and $\frac{CI^2}{BC^2}$ being given)

$QG : QH :: \frac{CI^2}{BC^2} \quad \frac{CI^2}{AC^2} :: AC^2 : BC^2$, therefore

$QF : QE \left(:: \frac{CI}{AC} \cdot QG : \frac{CI}{BC} \cdot QH\right) :: \frac{CI}{AC} \cdot AC^2 : \frac{CI}{BC} \cdot BC^2 ::$

$AC : BC$,

and by proceeding in a ſimilar manner with QE and QD or QF and QD we ſhall get

$QE : QD :: BC : BA$ or $QF : QD :: AC : AB$.

Hence the point Q may be found, by Prob. XXX. Simp. on Conſtr. of Geom. Probs. at the end of his Geom. the perpendiculars (when the ſum of their ſquares is a minimum) being as the ſides on which they fall.

Now twice the area of the triangle is

$AB \times QD + BC \times QE + AC \times QF$, but

$AB : BC :: QD : QE = \frac{BC}{AB} \cdot QD$, and

$AB : AC :: QD : QF = \frac{AC}{AB} \cdot QD$, whence

twice

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twice the area $= AB \times QD + BC \times \frac{BC}{AB} \cdot QD + AC \times \frac{AC}{AB} \cdot QD$

$$= \frac{QD}{AB} \times (AB^2 + BC^2 + AC^2),$$

and the ſum of the ſquares of the perpendiculars

$$= QD^2 + \frac{BC^2}{AB^2} \cdot QD^2 + \frac{AC^2}{AB^2} \cdot QD^2$$

$$= \frac{QD^2}{AB^2} \times AB^2 + BC^2 + AC^2); \text{ hence}$$

$$AB^2 + BC^2 + AC^2 : \frac{QD}{AB} (AB^2 + BC^2 + AC^2)$$

$$:: \frac{QD}{AB} \times (AB^2 + BC^2 + AC^2) : \frac{QD^2}{AB^2} \times (AB^2 + BC^2 + AC^2).$$

Q. E. D.

The ſame, by Mr. W. Smith.

LEMMA. Fig. 663, 664, pl. 32. If the baſe AB of a $\triangle$ be divided in Q ſo that $AQ : QB :: AC^2 : BC^2$ the ſum of the ſquares of the $\perp$s demitted from Q on the ſides will be leſs than if Q were taken any where elſe in the baſe.

DEMONSTRATION. Draw QL ſo that the $\angle ALQ = \angle ABC$; and demit the $\perp$s QR, QP then by conſtruction $AQ : QB :: AQ^2 : AQ \cdot QB$ (by hypotheſis) $:: AC^2 : CB^2$ (by ſim. $\triangle$s ACB, AQL) $:: AQ^2 : QL^2$; therefore $AQ \cdot QB = QL^2$. Now if QT be always erected $\perp$ to AB and equal to QL, the locus of T will be a right line given in poſition, as AT but by reaſon of the equal $\angle$s L and B in the right angled $\triangle$s QRL, QBP, $QR^2 + QP^2$ will be a minimum when $QL^2 + QB^2$ is ſuch, that is, when $(QT^2 + QB^2)$ BT^2 or BT is ſuch, and it has been proved above that $AQ \cdot QB = (QL^2)$ QT^2; therefore the $\angle$ ATB is right and BT or BT^2 or $BQ^2 + QL^2$ or $QP^2 + QR^2$ is a minimum.

Q. E. D.

Hence it appears that with reſpect to the $\triangle$ ACB the locus of the point Q mentioned in the queſtion is the right line CQ and the

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the point itſelf from which the ⊥s demitted on the ſides have the ſum of their ſquares, a minimum muſt be a point q ſuch that drawing $MqN \parallel AB$, $MN \cdot Nq$ may be $= BN \cdot NC$, for demitting the ⊥s ql, qE, qF and drawing $qO \parallel QL$ and $qS \parallel CB$ the $\triangle$s qOF, qNE, qSI are all ſimilar, therefore $qO^2 + qN^2 + qS^2$ is a minimum, that is to ſay $Nq \cdot NM + NB^2$ is a minimum; now it appears (ſince $Nq \cdot NM : NC^2$ in a given ratio) that if NG be erected ⊥ to NC and ſuch that $NG^2 = NM \cdot Nq$ the locus of the point G will be a right line CG given in poſition and GB^2 will always be equal to $BN^2 + NG^2 = BN^2 + NM \cdot Nq = BN^2 + Nq^2 + qO^2 = qS^2 + qN^2 + qO^2$, therefore $qS^2 + qN^2 + qO^2$, and conſequently $qr^2 + ql^2 + qE^2$ will be a minimum when BG is a minimum, that is when BG is ⊥ to CG or as above ſaid when $BN \cdot NC = (NG^2) = NM \cdot Nq$.—Hence the following

Demonstration of the Theorem.

Produce AB to Y ſo that $AB \cdot AY$ may be $=$ to $AB^2 + BC^2 + CA^2$, divide AB in q ſo that $Aq : qB :: AC^2 : CB^2$, join Cq and demit the ⊥CD; draw $MQNG \parallel AB$ ſo that $MN \cdot NQ = BN \cdot NC$, draw Bd, Bq', $Ba \parallel CD$, Cq, CA reſpectively, join XY meeting BD in g and draw $ngQ'm \parallel ba$. Then from the conſtruction it is evident that $AB \cdot BY = AC^2 + CB^2$ and alſo (ſince $Aq : qB :: AC^2 : BC^2$) that $Aq \cdot BY = AC^2$ and $Bq \cdot BY = BC^2$; now $YB : dX$ $(:: Bg : gd :: Bn : nC) :: Bn^2 : Bn.nC :: YB \cdot Bq : AB \cdot (dX) \cdot Bq$ $:: BC^2 : aC \cdot Cq :: Bn^2 : mn.nq'$ that is $Bn^2 : Bn.nC :: Bn^2 : mn\ nq$; whence $Bn.nC = mn.nq$; therefore $ag = DG$. But $AY \cdot AB : AX \cdot AB :: dX : (dg \cdot AX)$ $DG \cdot DC$, that is $AC^2 + CB^2 + BA^2$: twice the area :: twice the area : the ſum of the ſquares of the ⊥s demitted from Q, for per the Lemma $qO^2 + qN^2 + qS^2 : qF^2 + qE^2 + ql^2 :: qS^2 : ql^2 :: BN^2 : DG^2 :: BN \cdot BC : DG \cdot DC\ (dg.AX)$; but $BN \cdot BC = qO^2 + qN^2 + qS^2$, therefore $DG \cdot DC = ql^2 + qE^2 + ql^2$.

N. B. In this ſolution reference muſt be had to both figures all along.

The ſame, by Mr. John Whitley.

Let ACB (fig. 697, pl. 33.) be any plane $\triangle$, and P a given point within it from which perpendiculars are drawn to the ſides. Then ſince the ſum of the ſquares of thoſe perpendiculars muſt be

a

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a minimum, any two of them may be varied independent of the third; therefore through the point P, draw GH parallel to BC meeting AC, and AB in H and G, and join AP, BP, CP, and GC. It will readily appear, that whenever the point P falls in GH, the ſum of the $\triangle$s APC, APB will always be equal to the conſtant $\triangle$ ACG, and conſequently if one of theſe $\triangle$s, as APB, be increaſed by the ſmall rectangle $AB \times m$, the other, APC, will have decreaſed by the ſame rectangle $AB \times m = AC \times n$; that is, if the perpendicular PD, be increaſed till it becomes equal to $PD + m$, the perpendicular PE will be decreaſed till it becomes equal to $PE - n$; and conſequently $PD^2 + PE^2$ will be ultimately equal to

$(PD + m) \times PD + (PE - n) \times PE$

and therefore $PD \times m = PE \times n$;

and becauſe $AB \times m = AC \times n$, we have $PD \times AC = PE \times AB$, or

$$PD = \frac{PE \times AB}{AC}.$$

In the very ſame way it may be ſhewn that

$$PF = \frac{PE \times BC}{AC}.$$

Now draw CR perpendicular to AB; then

$$CR \times AB = PD \times AB + PF \times BC + PE \times AC$$

$$= \frac{PE \times AB^2}{AC} + \frac{PE \times BC^2}{AC} + PE \times AC;$$

conſequently $PE = \frac{AC \times CR \times AB}{AB^2 + BC^2 + AC^2}$; wherefore

$$PF = \frac{BC \times CR \times AB}{AB^2 + BC^2 + AC^2};$$

and $PD = \frac{AB \times CR \times AB}{AB^2 + BC^2 + AC^2}$.

Hence $PE^2 + PF^2 + PD^2 = \frac{CR^2 \times AB^2}{AB^2 + BC^2 + AC^2}$; or

$AB^2 + BC^2 + AC^2 : CR \cdot AB :: CR \cdot AB : PE^2 + PF^2 + PD^2$,

that is,

$AB^2 + BC^2 + AC^2 : 2\triangle ACB :: 2\triangle ACB : PE^2 + PF^2 + PD^2$.

Q. E. D.

The

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The ſame, by Mr. James Cunliffe.

Fig. 698, pl. 33. Let Q be the point in the triangle ABC from whence perpendiculars being drawn to the ſides, the ſum of their ſquares will be the leaſt poſſible. Through Q let the right lines DE, FG, and HI be drawn parallel to the ſides AB, BC, and AC reſpectively.

Put $AB = a$, $BC = b$, $AC = c$, and $HF = x$, and let A denote the area of the $\triangle$ ABC. Then $\frac{2A}{AB}$ will expreſs the length of a pependicular from the $\angle$ C upon AB; and, by the property of the point Q, as demonſtrated in the Tracts and Selections, in Art. 28 of Playfair's Paper on the Origin and inveſtigation of Poriſms.

$AB^2 : AC^2 :: FQ : QG :: HF : DQ = AH$, that is,

$AH = \frac{AC^2 \times HF}{AB^2} = \frac{c^2x}{a^2}$; and for the ſame reaſon

$AB^2 : BC^2 :: HF : QE = FB = \frac{BC^2 \times HF}{AB^2} = \frac{b^2x}{a^2}$;

therefore $AH + HF + FB = \frac{c^2x}{a^2} + x + \frac{b^2x}{a^2} = AB = a$;

Whence $x = \frac{a^2}{a^2 + b^2 + c^2}$. Again per ſimilar $\triangle$s

$AB : HF :: \frac{2A}{AB} : \frac{2A \times HF}{AB^2} = \frac{2Ax}{a^2} = \frac{2Aa}{a^2 + b^2 + c^2} =$

the length of a perpendicular from Q upon AB. And in the ſame way we ſhall find that

$\frac{2Ab}{a^2 + b^2 + c^2}$ and $\frac{2Ac}{a^2 + b^2 + c^2}$ expreſs the lengths of perpendiculars from Q, upon BC and AC reſpectively.

And the ſum of the ſquares of the ſaid perpendiculars is

$\frac{4A^2 \times (a^2 + b^2 + c^2)}{(a^2 + b^2 + c^2)^2} = \frac{4A^2}{a^2 + b^2 + c^2}$ from whence it is very obvious that twice the area of the $\triangle$ ABC is a mean proportional

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portional between the ſum of the ſquares of the perpendiculars from Q, and the ſum of the ſquares of the ſides of the triangle.

The ſame, anſwered by Amicus.

LEMMA. *Problem.* In a plain triangle ACB (fig. 662, pl. 32.) it is required to find a point Q in one ſide, ſuch that the ſum of the nth powers of QF and QE, drawn perpendicular to the other two ſides, may be a minimum.

Put $QG = x$, $GH = a$; then $QF = x \times$ fin. G, and $QE = (a - x) \times$ fin. H; and the quantity to be a minimum is $x^n \times$ fin.^{n}G $+ (a - x)^n \times$ fin.^{n}H; putting the fluxion $= o$ we eaſily get $x^{n-1} \times$ fin.^{n}G $= (a - x)^{n-1} \times$ fin.^{n}H; that is $QG^{n-1} \times$ fin.^{n}G $= QH^{n-1} \times$ fin.^{n}H. Whence $QG^{n-1} : QH^{n-1} ::$ fin.^{n}H : fin.^{n}G $::$ GC^n : CH^n. Therefore the point Q is found by dividing GH in a certain given proportion.

Cor. The $(n - 1)$th powers of the perpendiculars QF and QE are as the ſides of the triangle on which they fall.

For, ſince x fin. G $=$ QF and $(a - x)$ fin. H $=$ QE, therefore, becauſe $x^n \times$ fin.^{n}G $= (a - x)^n \times$ fin.^{n}H, $QF^{n-1} \times$ fin. G $= QE^{n-1} \times$ fin. H. Conſequently $QF^{n-1} : QE^{n-1} ::$ fin. H : fin. G $::$ GC : CH.

Demonſtration of the Prize Queſtion.

Let ACB be the triangle, and Q the point ſuch that the ſum of the ſquares of QF, QE, and QD is a minimum; and through Q draw GH parallel to one ſide as AB. Then, becauſe all the perpendiculars drawn from GH to AB are equal, the ſum of the ſquares of QF and QE muſt be leſs than the ſum of the ſquares of perpendiculars drawn from any other point in GH to GC and CH. Therefore (Cor. to Lem. putting $n = 2$) QF : QE $::$ GC : CH $::$ AC : CB.

But what has been demoſtrated of QF, QE, and the ſides AC and CB may in like manner be demonſtrated of QF, QD and AC, AB, therefore we have theſe two proportions.

QF : QE :: AC : CB
QF : QD :: AC : AB.

Now,

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Now, from the proportions juſt found, we readily derive theſe two (E. 1. 6)

$$QF^2 : QE^2 :: AC \times QF : BC \times QE$$
$$QF^2 : QD^2 :: AC \times QF : AB \times QD$$

whence (E. 12. 5)

$$\{QF^2 : (QF^2 + QE^2 + QD^2) :: AC \cdot QF : (AC \cdot QF + BC \cdot QE + AB \cdot QD =) 2\triangle ACB.\}$$

Again, from the ſame two proportions, the two following are in like manner derived, viz.

$$QF \times AC : QE \times BC :: AC^2 : CB^2$$
$$QF \times AC : QD \times AB :: AC^2 : AB^2$$

whence (E. 12. 5)

$$\{QF \times AC : (QF \cdot AC + QE \cdot BC + QD \cdot AB =) 2\triangle ACB :: AC^2 : (AC^2 + BC^2 + AB^2)\}$$

But, if the two proportions that have now been demonſtrated be taken alternately, the two firſt ratios will evidently be equal; therefore alſo the two latter ratios will be equal; that is

$$QF^2 + QE^2 + QD^2 : 2\triangle ACB :: 2\triangle ACB : AC^2 + AB^2 + BC^2.$$

Q. E. D.

The Medal for ſolving the Mathematical Prize Queſtion is decided in favour of Mr. I. T. M'Donell, who will pleaſe to ſend for it to Mr. Glendinning's.

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ARTICLE XXI.

Anſwers to the Mathematical Queſtions propoſed in ARTICLE XV. No. XIII.

I. QUESTION 301, *anſwered by Mr.* Tho. Boole.

The ſide of the octagon being 10 chains its area was 48·28427 acres, and annual rent £96·56854. Therefore, by Dr. Hutton's Theorems for Annuities in arrear, (at page 260, Vol. I. of he Courſe.)

$$\frac{1{\cdot}05^{50}-1}{1{\cdot}05-1}\times 96{\cdot}56854 = £20216{\cdot}8s.\ 7\tfrac{1}{2}d.$$ the ſum required.

Ingenious anſwers were alſo given by Meſſrs. Amos, Collins, Cunliffe, Exton, Francis, Hearding, Morley, I. Smith, Squire, Thompſon, Tyro Philomatheticus, *and* Whitley.

II. QUESTION 320, *anſwered by Maſters* Wm. Amos *and* Wm. Exton, *at Frieſton School, near Boſton.*

Put a = axis, s = ſurface, S = ſolidity, and $p = 3{\cdot}141592$ &c. Then

Firſt, when $s = S$, we have $pa^2 = \frac{1}{6}pa^3$, or $a = 6$;

Secondly, when $a = S$, we have $a = \frac{1}{6}pa^3$, or $s = pa^2 = 6$; and

Thirdly, when $a = s$, we have $a = pa^2$, or $a = \frac{1}{p}$, and

therefore, $S = \frac{1}{6}pa^3 = \frac{1}{6p^2} = {\cdot}016885$.

That is, when the ſuperficies and ſolidity of a globe are equal the axis is 6; when the axis and ſolidity are equal the ſuperficies is 6; and when the axis and ſuperficies are equal, the ſolidity is equal

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equal to unity divided by 6 times the ſquare of the circumference of a circle whoſe diameter is unity or 1.

Ingenious anſwers were alſo given by Meſſrs. Boole, Collins, Cunliffe, Francis, Hearding, Morley, Thompſon, Tyro Philomatheticus, *and* Whitley.

III. QUESTION 303, *anſwered by Mr.* Piſhey Thompſon, *of Frieſton School, near Boſton.*

Firſt 70½, the cubic inches in an ale quart, divided by $(4^3 \times .7854) = 5.612 = AC$, (fig. 696, pl. 33.) Since ad is perpend. to Cac; by Euc. 13, VI. $Ca \times ac = (ad)^2$, whence $Ca = .268$. Let ae be drawn parallel to CA; then will $eb = AB - ae - bB = AB - 2Ae = AB - 2Ca = 3.464$; finally $(ab)^2 = (ae)^2 + (eb)^2 = (CA)^2 + (eb)^2 = 43.494544$; or $ab = 6.595$ inches nearly the length of the braſs plate.

The ſame, by Mr. Thos. Morley.

Firſt $\frac{70.5}{4^2 \times .7854} = 5.6102$ the depth of the quart. Then $2\sqrt{(2^2 - 1^2)} = 3.46414$, the diſtance of the lower end of the plate from a point on the bottom directly under the upper end; therefore $\sqrt{(5.6102^2 + 3.4641^2)} = 6.5935$ inches the length of the plate required.

Other ingenious ſolutions were given by Meſſrs. Collins, Cunliffe, Francis, Hearding, I. Smith, Squire, Tyro Philomatheticus, *and* Whitley.

IV. QUESTION 304, *anſwered by Mr.* Tho. Boole.

Fig. 686, pl. 33. Let AB and CD be the poles, EF the chord, G the point in which the poles meet, and GH the perpendicular on the ground, then will ILK be the poſition of the chord when the poles meet;

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Now IL = LK = HD = DK = 5 yards, and $\sqrt{(5^2 + 3^2)} = \sqrt{34} = DL$;
and DL : radius :: HL : fine $\angle$ HDL = 30° 57′ 49″,
alſo KL and KD being equal we have
DK : radius :: $\frac{1}{2}$DL : cofine $\angle$ LDK = 54° 19′ 53″,
and HDL + LDK = HDG = 85° 17′ 42″,
and cofine $\angle$ HDG : HD :: radius : DG = 60·9567 yards, the length of the poles, as required.

The ſame anſwered, by Mr. Tho. Squire, *Baldock.*

Let AB, CD repreſent the poles, EF the rope, *e* the middle point thereof, or the firſt poſition of the weight.

Let L be the poſition of the weight when the poles meet at G, biſect BD in H and join HG, DL, and draw LX perpend. to DL meeting DG in X, alſo draw VX parallel and PX perpend. to DH.

Then becauſe LK = *e*F = $\frac{1}{2}$BD = 5 and DF = DK = 5, we have LK = DK = KX = 5 (becauſe $\angle$ DLX is a right one); therefore DX = 10, and ſince Le = 2, HL = 2; therefore
DL = $\sqrt{(HL^2 + DH^2)} = \sqrt{(3^2 + 5^2)} = \sqrt{34}$, and
LX = $\sqrt{(DX^2 - DL^2)} = \sqrt{66}$. But by ſim. $\triangle$s,

$$DL(=\sqrt{34}) : HL(=3) :: LX(=\sqrt{66}) : VX = 3\sqrt{\frac{66}{34}} = 4{\cdot}179783;$$

alſo DP = DH − VX (= 8·20217) : DX (= 10) : DH (= 5) : BD
alſo DP = DH − VX : DX :: DH : DG = 60·95944 yards the length of the poles.

The ſame, by Mr. William Francis, *Taplow, near Maidenhead.*

Here $\sqrt{(5^2 - 2^2)} = Ie$, and $Ie \times 2 = IK$;
alſo $\frac{1}{2}$(BD − IK) = B*i*; and B*i* : BI :: BH : EG = DG = 60·96 yards nearly, the anſwer.

This Queſtion was alſo anſwered by Meſſrs. Collins, Cunliffe, Hearding, Morley, Tyro Philomatheticus, *and* Whitley.

V.

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V. QUESTION 305, *anſwered by Mr.* John Collins, *Kenſington.*

Let $p =$ the product of the required numbers, then the ſum of the nth powers will be expreſſed by

$s^n - n\,(s^{n-2})\,p + n.\,\frac{n-3}{2}\,(s^{n-4})\,p^2 - n.\,\frac{n-4}{2}\,.\,\frac{n-5}{3}\,(s^{n-6})\,p^3$

$+ n.\frac{n-5}{2}.\,\frac{n-6}{3}.\,\frac{n-7}{4}\,(s^{n-8})\,p^4 -$ &c. taking $n = 9$ we have $512 - 1152p + 864p^2 - 240p^3 + 18p^4 = 32$, or

$p^4 - \frac{40}{3}p^3 + 48p^2 - 64p + \frac{80}{3} = 0$;

and this equation reſolved into its factors, becomes

$(p-2) \times (p-2) \times (p^2 - \frac{28}{3}p + \frac{20}{3}) = 0$, and conſeq.

$p^2 - \frac{28}{3}p + \frac{20}{3} = 0$; a quadratic, one root of which is

$\frac{14 - 2\sqrt{34}}{3} = p$. Now from the ſquare of $(x+y)$ take $4p$; and extracting the ſquare root we have $x - y = \sqrt{(s^2 - 4p)}$.

Therefore $x = \frac{s + \sqrt{(s^2 - 4p)}}{2}$ and $y = \frac{s - \sqrt{(s^2 - 4p)}}{2}$.

Whence $x = 1{\cdot}4697175$ and $y = {\cdot}5302824$.

Solutions to this queſtion were alſo received from Meſſrs. Morley, *and* Whitley.

VI. QUESTION 306, *anſwered by Mr.* J. H. Hearding.

It is well known that a column of water whoſe height is 33 feet, is equal in weight to a column of air of the ſame baſe, and whoſe altitude is equal to that of the atmoſphere above the earth's ſurface. It has alſo been found by divers experiments, that a body of water is to a body of air of an equal bulk as 850 to 1. If any number

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number of altitudes of the atmosphere be taken in an increasing arithmetical progression, the respective densities of the air, at those altitudes, will form a decreasing geometrical progression; or, in other words, the densities of the air at the earth's surface B, and the heights BD, BF (see fig. 685, pl. 33) are to one another as the ordinates AB, CD, EF of the logarithmic curve ACE, whose asymptote is BDF. This is demonstrated by Dr. Halley, in the Philosophical Transactions, No. 181; it has also been very clearly proved by Dr. Gregory, in his valuable Treatise of Astronomy, Prop. 3, Book V. To apply what is here premised to the solution of the proposed question; assume AB = 33 feet, the height of a column of water; BD = 850, the altitude of a column of the atmosphere of equal weight with a column of water of 1 foot high; CD = 32 the density of the air at the height of 850 feet; BF = 30 miles = 158400 feet, the given altitude, and EF = the density of the air at that height; then, since in the present case, AB is supposed to be equal 1, CD will be represented by $\frac{32}{33}$, and,

as 850 (= BD) : 158400 (= BF) or
as 17 : 3168 :: log. (AB÷DC) : log. (AB÷EF)=2·4904021,
consequently log. EF = — 3·5095979, and EF = 0·0032329, the density of the air at the height of 30 miles when the density at the surface is = 1; let this = d, and put the capacity of the barrel and receiver together = 11 = c, that of the receiver alone = 9 = r, and the number of strokes of the piston, necessary to rarify the air to the given density d, = x; then, by Hutton's Course Vol. II. page 329,

$$x \text{ is} = \frac{\text{Log. } d}{\text{Log. } r - \text{Log. } c} = \frac{-2{\cdot}4904021}{-0{\cdot}0871502} = 28{\cdot}576 \text{ turns of}$$

the piston.

The same, by Mr. John Smith, *Alton Park.*

First, $\frac{30 \times 5280}{63551} = 2{\cdot}4924863$ the logarithm of the rarity of the air at the altitude of 30 miles, when the temperature is 55°.

Then

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Then $\frac{2\cdot4924863}{\log.(9+2)-\log.(9)} = 28\cdot6$, nearly the number of ſtrokes required.

See Dr. Hutton's Courſe of Mathematics, Vol. II. art. 329 and 341.

Ingenious ſolutions were alſo given by Meſſrs. Collins, Cunliffe, Thompſon, *and* Tyro Philomatheticus.

VII. QUESTION 307, *anſwered by Mr.* Francis, *the Propoſer.*

Draw the equilateral triangle FBC. (fig. 687, pl. 33.) On FC take FG = 12, and draw GH || to BC. From the centres G and H, with radii 8 and 9 deſcribe arcs interſecting in I, and draw GI. Then if ∠ FCD be made = ∠ FIG, the line CD will meet FI produced in the required point D.

For ſince the ∠s FCD, FIG are equal and GFI common to both the △s CDF, FIG, the other ∠s muſt alſo be equal and the △s ſimilar. If HI be drawn it may be proved in the ſame manner that the △s FIH, FBD are ſimilar.

Again, draw DK ⊥ and = to BD; biſect KC in L and produce LC to M, making LM = DL. From M, through D draw the line MA, on which demit ⊥s from B and C, then will ABCEDA be the garden required. For if KN be drawn ⊥ to AM the △s ABD, DKN, will be ſimilar and equal.

The △ EMC is alſo ſimilar to the △MKN;
EM is evidently = DN and MN = DE.

Hence as EM : EC :: MN : KN conſequently
$\frac{1}{2}$ DE·EC = $\frac{1}{2}$DN·KN = $\frac{1}{2}$AB·AD.

W. W. R.

Calculation. In the △ HIG are given all the ſides, whence the ∠s are = 41° 25′, 48° 11′, and 90° 24′. And the ∠ IGF = 108° 11′, the ∠ IHF = 101° 25′, ∠ IFG = 27° 40′, ∠ FIG = 44° 19′, ∠ IFH = 32° 20′, ∠ FIH = 46° 15′, and IF = 16·368. Therefore by ſim. △s, as IF (= 16·368) : FC :: IG : DC = 29·32 and :: HI : BD = 32·985. Now we have all the ſides of △BCD, conſequently the ∠s 14° 43′, 16° 36′, and 148° 41′. Hence ∠ KDC = 58° 41′, ∠ DKC = 54° 41′. and ∠ DCK = 66° 38′.

Alſo the ſide KC = 30·697, the ∠DLC = 82° 8′, DL = LM

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LM = 27.171, whence CM = 11.825; alſo $\frac{1}{2}$ ∠ DLC = ∠ CEM = 41° 4′, whence ∠ LDM = LMD = 48° 56′.

And therefore EC = 8.9154, and ED = 27 9316, and the area of the garden = 500 ſquare yards.

VIII. QUESTION 308, *anſwered by Mr.* Wm. Marrat.

Put .7854 = p, and let the conjugate axe be denoted by x, then $1000 - x$ is the tranſverſe axe, and $1000px - px^2$ is the area of the ellipſe; alſo $\frac{1}{40}x + 500$ is its value in guineas: therefore

$1000px - px^2 : \frac{1}{40}x + 500 :: 4840 : \frac{(\frac{1}{40}x + 500) \times 4840}{1000px - px^2} =$

the value of one acre, which by the queſtion is to be a minimum,

or $\frac{x + 1000}{1000x - x^2}$, a minimum.

This put into fluxions and reduced, becomes

$x^2 + 2000x = 1000$; whence $x = 414.21356824$ the conjugate and $1000 - x = 585.7864376$ the tranſverſe axis. Hence the area is 3gac. 1r. 20p. nearly.

The ſame, anſwered by Tyro Philomatheticus.

Let x = tranſverſe, y = conjugate, a = 1000 yards, and p = .7854;

Then pxy = the content in ſquare yards,

and $\frac{1}{40}x + y$ = the value in guineas,

therefore $\frac{\frac{1}{40}x + y}{pxy}$ = the price per yard, which by the queſtion ſhould be a minimum, or $\frac{x + 2y}{xy}$ = a minimum.

In fluxions $\frac{xy(\dot{x} + 2\dot{y}) - (x + 2y)(x\dot{y} + \dot{x}y)}{x^2y^2} = 0$,

or $x^2\dot{y} + 2y^2\dot{x} = 0$, but $\dot{x} = -\dot{y}$, hence $x^2 = 2y^2$, or

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$x = y\sqrt{2}$, and therefore $x = a \times \frac{\sqrt{2}}{1+\sqrt{2}} = 585{\cdot}79$, $y = a \times \frac{1}{1+\sqrt{2}} = 414{\cdot}21$, and the area $= 190569\,456$ yards.

IX. QUESTION 309, *anſwered by Mr.* John Smith.

The ſurface of a ſphere or hemiſphere is equal to the curve ſurface of its circumſcribing cylinder, therefore $3{\cdot}1416 \times 24 \times 40 = 3015{\cdot}936$ feet $= 335{\cdot}104$ yards, the ſurface of the balloon, or the quantity of ſilk required.

The ſolidity of a ſphere being $\frac{2}{3}$ that of its circumſcribing cylinder $\cdot 7854 \times 23^2 \times 32 - 14476{\cdot}4928$ feet the capacity of the balloon.

Hence $14476{\cdot}4928 \times (1\frac{2}{9} - \frac{1\frac{2}{9}}{13}) = 16332{\cdot}4534$ ounces $=$ $9{\cdot}11409$ cwt. the weight with which it would aſcend.

Now by Dr. Hutton's Menſuration, Prob. 2, page 269, 3rd. Edit. $(8800)^2 \div 3000 = 25813\frac{1}{3}$ the parameter, and, by Rule 1, Prob. 4. p. 271, ibid

$$\frac{25813\frac{1}{3}}{2} \times \left(\left\{ \sqrt{\left[\left(\frac{17600}{25813\frac{1}{3}}\right)^2 + 1 \right]} \right\} \times \frac{17600}{25813\frac{1}{3}} + \text{hyp.log.} \left\{ \frac{17600}{25813\frac{1}{3}} + \sqrt{\left[\left(\frac{17600}{25813\frac{1}{3}}\right)^2 + 1 \right]} \right\} \right) =$$

$18954{\cdot}6502$ yards $= 10{\cdot}76968$ miles, the length of the parabolic tract deſcribed by the Aeronaut.

It was alſo ingeniouſly anſwered by Meſſrs. Boole, Cunliffe, Francis, Hearding, Thompſon, *and* Squire.

X. QUESTION 310, *anſwered by* Ztrepmog.

An n ſhould have been put within the firſt radical of the firſt and third expreſſions, and an m in that of the ſecond expreſſion. This being corrected, put x for the value and alſo in that part of the ſurd

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ſurd where the expreſſion recurs inſtead of continuing the ſurd, which will give an equation for x.

Thus in the firſt we have

$$x = \sqrt[n]{a + \frac{b}{\sqrt[n]{a} + \&c.}} = \sqrt[n]{a + \frac{b}{x}}, \text{ therefore}$$

$$x^n = a + \frac{b}{x} \text{ and } x^{n+1} - ax = b.$$

In the ſecond we have

$$x = \sqrt[m]{a + \frac{b}{\sqrt[n]{c + \frac{d}{\sqrt[n]{a} + \&c.}}}} = \sqrt[m]{a + \frac{b}{\sqrt[n]{c + \frac{d}{x}}}}$$

$$= \sqrt[m]{a + \frac{bx^{\frac{1}{n}}}{\sqrt[n]{cx + d}}} = \sqrt[m]{\frac{a\sqrt[n]{cx + d} + bx^{\frac{1}{n}}}{\sqrt[n]{cx + d}}}.$$

Therefore $x^m \times \sqrt[n]{cx + d} = a \times \sqrt[n]{cx + d} + bx^{\frac{1}{n}}$.

And in the third we have

$$x = \sqrt[n]{\frac{a}{b + \frac{\sqrt[n]{a}}{b + \&c.}}} = \sqrt[n]{\frac{a}{b + x}}; \text{ therefore } x^{n+1} + bx^n = a$$

The chief uſe of theſe recurring ſurds or fractions is for ſolving equations which they ſometimes do with advantage.

Thus if $x^{n+1} + bx^n = a$, be an equation in which b is greater than a we have

$$x = \sqrt[n]{\frac{a}{b + x}} = \sqrt[n]{\frac{a}{b + \sqrt[n]{\frac{a}{b + x}}}} = \sqrt[n]{\frac{a}{b + \sqrt[n]{\frac{a}{b + \sqrt[n]{\frac{a}{b + \&c.}}}}}}$$

The ſame as above, which ſurd will continually approximate to the value of x.

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XI. QUESTION 311, *answered by* Cap. Geo. Gorry.

Let SCRH (fig. 699, pl. 33.) represent a section of the given sphere passing through the centre O, and ABGI the inscribed cube; then it is manifest that AI the diagonal of the cube is a diameter of the sphere passing through the centre O. Hence to find *a*I or AG, let it be considered that

$AI^2 = AG^2 + GI^2 = AG^2 + Ga^2 + aI^2 = 3AG^2$; therefore $AI = AG\sqrt{3}$, and $AG = AI \div \sqrt{3} = \frac{1}{3}AI\sqrt{3}$; that is, in the present case, the side of the inscribed cube is $4\sqrt{3} = 6{\cdot}928203230$. Next to find the side *cb*, *bd*, or *de*, of the base of the octagonal pyramid (whose vertex will be V) let it be denoted by x; and since *abc* is a right angled triangle having $ab = ac$,

it will be $ab : x :: 1 : \sqrt{2}$,

consequently $ab = x \div \sqrt{2} = \frac{1}{2}x\sqrt{2}$

and $ab + bd + dI = aI$;

that is $x + 2 \times \frac{1}{2}x\sqrt{2} = x + x\sqrt{2} = aI$;

hence $x = \frac{aI}{1+\sqrt{2}}$, and $x^2 = \frac{aI^2}{3+\sqrt{2}} = \frac{48}{3+\sqrt{2}}$ = square of a side of the octagonal base.

The solidity of the pyramid will therefore be

$$\frac{48}{3+\sqrt{2}} \times \frac{4\sqrt{3}}{3} \times 4{\cdot}8284271 = 91{\cdot}8322 \text{ &c.}$$

It remains to find the solidity of the greatest circular spindle, inscribed in the space between Aa, and SCR. Here $CD = OC - OD = 2{\cdot}535898384$, and $CE = 1{\cdot}26794919$; from which, since CH is known, we easily find the area of the generating segment $RCSE = 6{\cdot}38135136$.

Again $ER = \sqrt{(CE \cdot EH)} = \sqrt{\left\{\frac{1}{2}DC \times (1\frac{1}{2}DC + aI)\right\}} = 3{\cdot}6888611$.

Therefore (Dr. Hutton's Mensuration, page 215, Ed. II.)

$$\left\{\tfrac{4}{3}ER^3 - (EO \times \text{area } CER)\right\} \times 4 \times 3{\cdot}141593 = 20{\cdot}5315 \text{ &c.}$$

solidity of the circular spindle RSCD. And consequently $91{\cdot}8322 - 20{\cdot}5315 = 71{\cdot}3007$ the difference required.

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The same, by Mr. John Smith.

By Dr. Hutton's Menſuration, page 186, 3d Edit. $4\sqrt{3} = 6{\cdot}9282032$ the ſide of the cube. Then (ibid. pa. 85.) the area of an octagon whoſe ſide is 1 is $4{\cdot}8284271$, which divided by 4, half the perimeter, gives $1{\cdot}2071068$ nearly for the perpendicular from the centre to the middle of one of its ſides; and $6{\cdot}9282032 \div 2 = 3{\cdot}4641016$, the perpendicular from the centre of the baſe of the octagonal pyramid to the middle of one of its ſides; whence by ſimilar figures,

$(1{\cdot}2071068)^2 : 4{\cdot}8284271 :: (3{\cdot}4641016)^2 : 39{\cdot}764502$ the area of the baſe; then

$39{\cdot}764502 \times 6{\cdot}9282032 \times \frac{1}{3} = 91{\cdot}832183$ the ſolidity of the pyramid.

Next, $3{\cdot}4641016 + \frac{6 - 3{\cdot}4641016}{2} = 4{\cdot}7320508$ the central diſtance of the ſpindal;

and $\sqrt{(6^2 - 4{\cdot}7320508^2)} = 3{\cdot}688861$ nearly, = half its length.

Then by Trigonometry,

$6 : 3{\cdot}688861 :: 1\text{ (rad.)} : {\cdot}6148101 =$ ſin. $37{\cdot}938121$ deg. therefore

$$\frac{37{\cdot}938121 \times {\cdot}0174533 \times 6}{2} - \frac{3{\cdot}688861 \times 4{\cdot}7320508}{2} =$$

$3{\cdot}190678$, half the area of the generating ſegment.

Hence, by the Treatiſe above referred to, Prob. 21, pa. 158,

$$\left\{\tfrac{1}{3} \times (3{\cdot}688861)^3 - (3{\cdot}190678 \times 4{\cdot}7320508)\right\} \times 4 \times 3{\cdot}1415926\overline{5} = 20{\cdot}531552$$ the ſolidity of the ſpindle.

Then $91{\cdot}832183 - 20{\cdot}531552 = 71{\cdot}300631$ the difference required.

It was ingeniouſly anſwered by Meſſrs. Collins, Cunliffe, Hearding, *and* Whitley.

XII. QUESTION 312, *anſwered by Mr.* John Whitley.

Let FCBADF (fig. 674, pl. 93.) repreſent the garden, and ſuppoſe that CED is the longeſt walk which can poſſibly be made therein parallel to AB. Then by property of the ſemicircle

FE

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FE : CE :: CE : BE and by a property of the parabola
DE^2 : AB^2 :: FE : BF; wherefore
DE^2 : CE^2 :: AB^2 : BE × BF = CB^2, or
BC : AB :: CE : DE; componendo
BC + AB : AB :: DC : DE.

Now AB is conſtant, and CD is a maximum; therefore (BC + AB) × DE is a maximum, which being fluxed and reduced gives

$$DE = \sqrt{AB^2 - \left\{[\sqrt{(AB^2+8FB^2)}-AB]\times\frac{AB}{4FB}\right\}^2} =$$

27·548; hence CE = 48·406, and therefore CD = 84 yards nearly.

XIII. QUESTION 313, *anſwered by Mr.* Cunliffe.

ANALYSIS. Fig. 665, pl. 32. Draw the radius BO and from the centre O draw O*m* perpendicular to AC, and join B*m*. O*m* biſects AC by Euc. III. 3. and
$2(Am)^2 + 2(Bm)^2 = AB^2 + BC^2$, by Simſon's Geometry, Theor. 11, Book II. = a given magnitude by the queſtion; therefore $(Am)^2 + (Bm)^2$ = a given magnitude.

Again $(Am)^2 + (Om)^2 = AO^2 = BO^2$ by Euc. I. 47, and taking the difference between this and the preceding gives
$(Bm)^2 - (Om)^2$ = a given magnitude, whence the locus of *m* in this caſe is known to be a right line perpendicular to BO. Now draw PO; ſince the angle P*m*O is a right angle the locus of *m* in this caſe is known to be the circle whoſe diameter is PO; wherefore the interſection of theſe loci will be the point *m*, through which the line PAC muſt be drawn.

The ſame, by Mr. John Whitley.

ANALYSIS. Fig. 676, 677, pl. 33. Let O be the centre of the given circle ABC, join PB, PO, CO and biſect PB in R. Suppoſe that PAC is really drawn as required.

Let OS be drawn perpendicular to PAC, and join RS.

Then by known properties

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$AC = CS$, $\frac{1}{2}AB^2 + \frac{1}{2}BC^2 = BS^2 + CS^2$, $OC^2 - OP^2 = CS^2 - PS^2$; wherefore $\frac{1}{2}AB^2 + \frac{1}{2}BC^2 + OP^2 - OC^2 = BS^2 + PS^2 = 2BR^2 + 2RS^2$; therefore $2RS^2 = \frac{1}{2}AB^2 + \frac{1}{2}BC^2 + OP^2 - OC^2 - 2BR^2 =$ a given space, which take equal to $2Q^2$, and we shall then have the following Construction.

Upon PO as a diameter describe the semicircle POS, to which from R, apply $RS = Q$; through S draw PAC meeting the circle in A and C, and the thing will be done, as is manifest from the Analysis.

The same, by Mr. Steele, *the Proposer.*

Analysis. Let PAC (fig. 678, pl. 39.) be supposed drawn as required. Join AB, BC; bisect AC in G, and join BG, QG, QA, QB, the centre of the circle being at Q.

Now $CB^2 + BA^2$ being given, its equal $2GB^2 + 2GA^2$, or $2GB^2 + 2QB^2 - 2QG^2$ will be given; and QA or QB being a given line, it follows that, $2GB^2 - 2QG^2$ or $GB^2 - QG^2 =$ a given space.

Demit upon BQ the perpendicular GE. Then. $GB^2 - GQ^2 = BE^2 - QE^2$, or $BE^2 - QE^2 =$ a given space.

Again, $BE^2 - QE^2 = (BE + QE) \times (BE - QE)$
$= (BQ + 2QE) \times BQ$
$= BQ^2 + 2QE \times QB$; consequently
$2QE \times QB =$ a given space, or QE a given line.

The point E is therefore given, and the lines BE, EG, given in position; that is, the point G is in a right line given in position; also, since PQ is given in length and position, and QG perpendicular to GP, it follows that the point G is in the circumference of the circle described upon the diameter PQ; and therefore the point G is given.

Hence the following Construction.

Join the given points P, B with the centre Q, and on each side of the point B take, in BQ, BR, BR′ each equal to the side of a square equal to the given space; in BQ, make $BQ : QR' = QR : QE$; draw EG perpendicular to EB, meeting a semicircle on PQ in G; and draw, through P and G, the required line PAC.

XIV.

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XIV. QUESTION 314, *anſwered by Mr.* Cunliffe.

CONSTRUCTION. Fig. 679, pl. 33. Draw the radius CF perpendicular to AB and join FP. Upon CB take CR ſuch that $2CP \times CR = FP^2$, and upon CR as a diameter deſcribe a ſemicircle in which apply ST perpendicular to CR ſuch that $4(ST)^2$ may be equal to the given ſum of the ſquares of the perpendiculars.

Again, upon CP as a diameter deſcribe another ſemicircle, and through O, the point where this latter ſemicircle cuts the line ST, draw the line required.

DEMONSTRATION. Draw CO which will be perpendicular to, and biſect DE by Euc. 31 and 3. III. alſo draw the perpendiculars D*m* and E*n*.

Becauſe of the ſimilar triangles CPO and COS,
$CP : CO :: CO : CS$, whence $CP \times CS = CO^2$.

Alſo by reaſon of the ſimilar triangles PD*m* and PE*n*,
$PD^2 : PE^2 :: Dm^2 : En^2$, and by compoſition
$PD^2 + PE^2 : PE^2 :: Dm^2 + En^2 : En^2$, alternately
$PD^2 + PE^2 : Dm^2 + En^2 :: PE^2 : En^2$
$:: CP^2 : CO^2 :: CP : CS$,
that is $CP : CS :: PD^2 + PE^2 : Dm^2 + En^2$.

Now $PD^2 + PE^2 = DE^2 - 2PD \times PE = 4DO^2 - 2PD \times PE$
$= 4DC^2 - 4CO^2 - 2PD \times PE$;
and $PD \times PE = AP \times PB = DC^2 - CP^2$, wherefore
$PD^2 + PE^2 = 4DC^2 - 4CO^2 - 2 \times (DC^2 - CP^2)$
$= 2DC^2 + 2CP^2 - 4CO^2$
$= 2DC^2 + 2CP^2 - 4CP \times CS$. And by conſtr.
$2CP \times CR = FP^2 = DC^2 + CP^2$, or
$4CP \times CR = 2DC^2 + 2CP^2$; whence
$PD^2 + PE^2 = 2DC^2 + 2CP^2 - 4CP \times CS$
$= 4CP \times CR - 4CP \times CS$
$= 4CP \times (CR - CS) = 4CP \times RS$. Therefore
$CP : CS :: PD^2 + PE^2 = 4CP \times RS : 4CS \times RS = 4ST^2$;
and from what is deduced above
$CP : CS :: PD^2 + PE^2 : Dm^2 + En^2$, wherefore
$Dm^2 + En^2 = 4ST^2$ the given ſpace by the conſtruction.

The following THEOREM may be gathered from the preceding ſolution.

In a given circle let the radius CF be drawn perpendicular to the diameter AB, and let the right line FP be drawn to any point P in AB; alſo let CR be taken a third proportional to twice

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CP and FP, and let circles be described upon CP and CR as diameters; then if at any point S between C and P, a right line SOT be drawn perpendicular to AB cutting the peripheries of the circles whose diameters are CP and CR, in O and T respectively; and if through P and O a right line be drawn to cut the given circle in D and E, and Dm, En be drawn perpendicular to AB then will $Dm^2 + En^2 = 4(ST)^2$.

XV. QUESTION 315, *answered by Mr.* Cunliffe.

Fig. 680. pl. 33. Let a circle be described about the triangle, and bisect the base AB by the perpendicular MN meeting the circumference of the circumscribing circle below the base in N, and draw NC. With the centre N and radius NB describe another circle cutting NC in E, which will be the centre of the circle inscribed in the triangle ABC as is very well known; wherefore from E demit Ea perpendicular to AB and it will be the radius of the inscribed circle, and a the point of contact thereof with the base AB. Produce CN till it cuts the circumference of the circle whose centre is N in F, and draw Fb perpendicular to AB.

Then because NE = NF, Mb must be equal to Ma, and therefore Bb = Aa. Again by Euc. 35, III.

LN × LC = AL × LB = LE × LF, whence

LN : LF :: LE : LC; and because of the ‖s NM, Eb,

LN : LF :: LM : Lb, therefore

LE : LC :: LM : Lb, wherefore Cb and EM being drawn, will be parallel to each other, from whence and what is before laid down the truth of the proposition will be manifest.

The same, by Mr. J. Collins, *Kensington.*

Produce ME to meet the perpendicular, let fall from the vertical angle upon the base, in P (fig. 681, pl. 33.)

It is evident that the triangles BMF and CDE are similar, and so are the triangles FME and CPE (DE being perpendicular to BC). Hence

BF (EF) : FM :: CE : CP :: CE : DE;

therefore CP = DE = Eg, and EgCP is a parallelogram;

wherefore

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wherefore ME and *bg*C are parallel, and
*b*M : M*a* :: E*g* : E*a*, and therefore *b*M = M*a*;
Wherefore B*b* + *b*M = M*a* + *a*A, that is, BM = MA.

Q. E. D.

The same, by Mr. John Dawes, *Birmingham.*

Fig. 637, pl. 31. Let fall the perpendiculars E*a*, CK, and *l*M, continue ME till it meets CK in *p*. Now it is well known that C*p* = E*a*; and because C*p*M*l* is a parallelogram C*p* = *l*M = E*a*; but B*a* = A*b*, and the triangles *b*lM, ME*a* are similar and equal, therefore M bisects *ba*, consequently it bisects AB, the base of the triangle.

Q. E. D.

The same, by Mr. Thomas Morley.

Fig. 681, pl. 33. C*b* being joined will meet the circumference of the inscribed circle in the same point as *a*E produced to the circumference, as at *g*, by Prop. 58, Student. Therefore, because *a*E = E*g*, and EM ‖ *bg*, *b*M = M*a*; consequently BM is equal MA.

Q. E. D.

The same, by Mr. Whitley.

Fig. 673, pl. 32. Bisect AB in M, and join C*a*, EM; produce ME to meet C*a* in P. Then because AM = BM, and A*a* (or AM + *a*M) = B*b* (or BM + *b*M), we have *a*M = *b*M. But it is now pretty well known that CP = *a*l' (see Gent. Diary, queſt. 5, 1801). Wherefore the two ſides C*a*, *ba* of the triangle C*ab* are biſected in the points P and M reſpectively; therefore PEM is parallel to C*b*; that is, EM drawn parallel to C*b* biſects the baſe AB in M.

This propoſition was alſo demonſtrated by Limenus, *the Propoſer.*

XVI.

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XVI. QUESTION 316, *answered by* Astronomicus.

It is manifeſt that the duration of twilight is different at the ſame time of the year, in different climates of the earth; theſe differences are clearly ſtated in a few words in Art. 107, of O. Gregory's aſtronomy. Thus, "If a place be ſituated in a *parallel ſphere*, or nearly ſo, the apparent motion of the ſun being either quite or nearly parallel, he will be carried round for ſome months at a leſs depreſſion than 18°, during which there will, of courſe, be no real night. In a *right ſphere*, the twilight is ſhorteſt, becauſe the ſun riſing and ſetting at right angles to the horizon, he is depreſſed 18° below the horizon in the ſhorteſt period. In any place in an *oblique ſphere*, the nearer it is to one of the poles, the longer the twilight; and conſequently, the nearer the equator, the ſhorter the twilight."

The twilight then being abſolutely a minimum at the equator, at all times of the year, it is eaſy to find its duration; for at the time of the equinoxes it is nothing more than to find in what time 18° are paſſed over by the ſun in a right ſphere, the whole circle or 360° being paſſed over in 24 hours; thus, as 360° : 18° :: 24h. : $\frac{24 \times 18}{360} = \frac{6}{5}$ = 1h. 12m. duration of ſhorteſt twilight at the equinoxes.

And at any other time of the year let P (fig. 666, pl. 32.) repreſent the pole, PH the horizon, being a ſecondary to the equator in a right ſphere, H the ſun's place, and HC = 18°, the diſtance from the horizon to the crepuſculum circle. Then PH = 90° — ſun's decl. and as H is a right angle we have, ſine PH : radius :: tang. HC : tang. P. This analogy when the ſun's decl. is 7° 9′, as it is, on Oct. 12th. gives 18° 8′ nearly for the angle P; and this converted to time gives 1h. 12m $\frac{8}{15}$. for the duration of the twilight at the equator on October 12th.

As to the duration of the twilight on the ſame day in lat. 52° 12′ N. it is given at Pa. 94, *O. Gregory's Aſtronomy*, and at Pa. 19, *Profeſſor Vince's Aſtronomy*; both of whom ſtate it at about 1h. 58m $\frac{1}{2}$. So that the duration of twilight at the equator (which is *abſolutely* the ſhorteſt twilight) is ſhorter than that in lat. 52° 12′ N. by 46m $\frac{1}{2}$.

Remark. Hence appears the miſtake of thoſe who conceive that the common analogy for the ſhorteſt twilight is applicable to queſtions like this:—for, although that analogy enables us to find the

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the sun's declination when the twilight is a *minimum* at any given latitude, yet the converse of the analogy will not assist us in finding the latitude *where* the twilight is a *minimum*, and the sun's declination given:—the problems are not one and convertible, but two and distinct.

XVII. QUESTION 317, *answered by Mr.* Cunliffe.

It is pretty obvious from what Mr. Simpson has delivered on the maxima and minima of geometrical quantities in the Scholium to Theorem 8, that the sides of the given triangle will be bisected at the points of contact of the greatest inscribed ellipsis. This being noticed, let ABC (fig. 654, pl. 22.) be a given triangle with its greatest inscribed ellipsis touching the sides AB, AC and BC in the middle points D, E, and F respectively; join EF and draw CD intersecting it in d; also draw AF intersecting CD in O. Because the sides AC, BC, are bisected in E and F, the line EF is parallel to AB, which is a tangent to the curve at D, and therefore EF is bisected in d; wherefore CD passes through the centre of the ellipsis, as appears from the properties of the Conic Sections. And for the same reason AF will pass through the centre of the ellipsis, and consequently O, the intersection of CD and AF will be that centre, and OD a semi-diameter. Now between O and C take OH = OD, then will DH be a diameter of the ellipsis.

And because of the parallels AB, EF,
CF : CB :: 1 : 2 :: dF = dE : AD :: Od : DO, that is,
Od = dH : DO :: 1 : 2, and by compos. Dd : DO :: 3 : 2;
but dH : DO :: 1 : 2, as just deduced,
whence Dd × dH : DO^2 :: 3 : 4.

Let the semi-diameter OG be drawn parallel to AB; then by the property of the ellipsis

$$D\delta \times \delta H : DO^2 :: \delta E^2 = \frac{AB^2}{4^2} : OG^2;$$ wherefore

$$3 : 4 :: \frac{AB^2}{4^2} : OG^2 = \frac{AB^2}{3 \times 4},$$ whence the semi-conjugate diameters are known and the ellipsis may be described by prob. 77. B. 1. Emerson's Conics.

The demonstration of the following THEOREM may be deduced from the foregoing solution.

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The sum of the squares of two conjugate diameters of the greatest ellipsis that can be inscribed in any plane triangle is equal to two-ninths of the sum of the squares of the three sides of the triangle.

For, $OD = \frac{1}{3}$ CD as is well known, and

$CD^2 = \frac{AC^2 + BC^2}{2} - \frac{AB^2}{4}$, by Simp. Geo. 11. II. whence

$OD^2 = \frac{CD^2}{9} = \frac{AC^2 + BC^2}{18} - \frac{AB^2}{36}$, and from what has been deduced

$OG^2 = \frac{AB^2}{18}$; by adding these together

$OD^2 + OG^2 = \frac{AC^2 + BC^2 + AB^2}{18}$, and therefore

$4OD^2 + 4OG^2 = \frac{2}{9} \times (AC^2 + BC^2 + AB^2)$.

Q. E. D.

The same, by Mr. John Whitley.

Let ACB (fig. 656, pl. 32.) be the given triangle, bisect AB in D, and join CD; suppose D*n*S*m* the greatest inscribed ellipsis; draw the conjugate diameter *om* and the tangent ST parallel to AB, draw also SE parallel to BC.

Now by similar triangles

DC : DB :: DS : DE, alternately

DC : DS :: DB : DE; DB^2 : EDB = (DB - ST) DB = DB^2 - DB·ST

But by Emerson's Conics, Prop. 47, on the ellipsis,

DB × ST = om^2; wherefore

DC : DS :: DB^2 : DB^2 - om^2 :: DC·*om* : DS·*om* = a maximum by the question. Consequently

DB^2·DS·*om* = (DB^2 - om^2)·DC·*om* will be a max. but DC is constant, therefore when DS·*om* is a maximum, the solid (DB^2 - om^2)·*om*, contained under one leg of a right angled triangle and the square of the other will be a maximum; which is the case when $om^2 = \frac{1}{3}$DB^2 (See Simpson's Geom. Theo. 18. max. et min.). Hence DS = $\frac{1}{3}$DC.

Now

Now let $s =$ fine of the angle CDB; then will

$DS \times nm \times s = \frac{2}{3}DC \times 2\sqrt{\frac{1}{3}}DB^2 \times s = \frac{4DC \cdot DB \cdot s}{9}\sqrt{3} =$ area,

and $nm^2 + DS^2 = \frac{4DB^2}{3} + \frac{4DC^2}{9} =$ fum of the ſquares of the ſides of a parallelogram circumſcribing the ellipſis $Dn\,Sm$; let the former be taken equal to $2A$ and latter equal to $4B$; alſo let $T =$ the tranſverſe and $C =$ the conjugate diameters of the ellipſis. Then per Conics, $T \times C = 2A$, and $T^2 + C^2 = 4B$; wherefore

$T = \sqrt{(B + A)} + \sqrt{(B - A)}$ and $C = \sqrt{(B+A)} - \sqrt{(B-A)}$, which are general Theorems for aſcertaining the principal diameters of the greateſt ellipſis, that can be inſcribed in a given triangle.

Cor. 1. If the ellipſis touch the ſides of the triangle in D, Q, and R, then will $AR = CR$, and $BQ = QC$.

Cor. 2. The point O which is the centre of the ellipſis is the centre of gravity of the triangle.

XVIII. QUESTION 318, *anſwered by Mr.* Cunliffe.

Let $a^2 - 1$, and $b^2 - 1$ denote the two numbers, for each of theſe when increaſed by 1 will manifeſtly be a ſquare; but their ſum and difference alſo when increaſed by 1 muſt be ſquares; that is $a^2 + b^2 - 1$, and $a^2 - b^2 + 1$ muſt each be a ſquare.

Put $a^2 + b^2 - 1 = c^2$, and $a^2 - b^2 + 1 = d^2$, and by adding theſe two together $2a^2 = c^2 + d^2$, from whence it appears that c^2, a^2, and d^2 are three ſquares in arithmetical proportion. Therefore we may put

$c = r \times (m^2 - n^2 + 2mn)$, $a = r \times (m^2 + n^2)$, and $d = r \times (n^2 - m^2 + 2mn)$; but $a^2 + b^2 - 1 = c^2$; whence $b^2 = c^2 - a^2 + 1 = 4r^2mn \times (m^2 - n^2) + 1$, by writing for c and a their foregoing values.

Aſſume $b = 2rs - r$, then

$b^2 = (2rs - 1)^2 = 4r^2s^2 - 4rs + 1 = 4r^2mn \times (m^2 - n^2) + 1$.

Whence

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Whence $r = \frac{s}{s^2 - mn \times (m^2 - n^2)}$, where s, m, and n may may be taken at pleasure.

Example I.

Take $m = 2$ and $n = 1$, then $r = \frac{s}{s^2 - 6}$, whence it is obvious enough, that if we take $s = 3$, then $r = 1$; and from what has been laid down
$a = r \times (m^2 + n^2) = 5$, and $b = 2rs - 1 = 5$, and hence
$a^2 - 1 = 24$, and $b^2 - 1 = 24$, in which case the two numbers are equal to each other.

Example II.

Take $m = 3$ and $n = 2$, then $r = \frac{s}{s^2 - mn \times (m^2 - n^2)} = \frac{s}{s^2 - 30}$ where it is pretty plain that if we take $s = 6$, then $r = 1$, and hence
$a = r \times (m^2 + n^2) = 13$, and $b = 2rs - 1 = 11$;
and hence again $a^2 - 1 = 168$ and $b^2 - 1 = 120$, being two numbers that will answer.

Example III.

Take $m = 4$ and $n = 3$, then $r = \frac{s}{s^2 - 84}$, where if s be taken $= 9$, then $r = -3$; and
$a = r \times (m^2 + n^2) = -75$ and $b = 2rs - 1 = -55$,
and hence $a^2 - 1 = 5624$ and $b^2 - 1 = 3024$, which are Mr. Bonnycastle's numbers.

Example IV.

Take $m = 5$ and $n = 3$, then $r = \frac{s}{s^2 - 240}$, whence it is not difficult to perceive, that if we take $s = 16$, then $r = 1$, and

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$a = r \times (m^2 + n^2) = 34$ and $b = 2rs - 1 = 31$;
and from these $a^2 - 1 = 1155$ and $b^2 - 1 = 960$, which are another pair of numbers that will answer.

Very general formulæ for whole numbers that will answer the question may be obtained as follows. We have before found that

$b^2 = 4r^2 mn \times (m^2 - n^2) + 1 = 4r^2 m^3 n - 4r^2 mn^3 + 1$

assume $b = 2r^2 mn^2 - 1$, then

$b^2 = (2r^2 mn^2 - 1)^2 = 4r^4 m^2 n^4 - 4r^2 mn^2 + 1$

$= 4r^2 n^2 m^2 n - 4r^2 mn^2 + 1.$

Whence $m = r^2 n^2$, $b = 2r^4 mn^2 - 1 = 2r^4 n^5 - 1$, and
$a = r \times (m^2 + n^2) = r \times (r^4 n^{10} + n^2) = rn^2 \times (r^4 n^8 + 1)$,
where r and n may be taken at pleasure.

Example I.

Suppose $r = 2$ and $n = 1$, then $a = 34$ and $b = 31$, which agrees with what was obtained in the fourth example, by the first method of solution.

Example II.

Suppose $n = 2$ and $r = 1$, then $a = 1028$ and $b = 511$, whence $a^2 - 1 = 1056783$ and $b^2 - 1 = 261120$, which are two other numbers that will answer.

Ingenious solutions were also received from Messrs. Boole, Collins, Hearding, *and* Whitley.

XIX. QUESTION 319, *answered by Mr.* Gregory.

Fig. 691, Pl. 33. Let Vv be a ray from the sun, touching V the vertex of the cone VABD, and v the extremity of the shadow. Let RGSH, RGsH, &c. be sections of the cone, whose axes RS, RS, &c. are parallel to Vv, and whose whole planes coincide with rays from the sun; these sections will manifestly be ellipses so long as the sun's alt. is less than the angle ABV of the

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cone, and in no other case can there be a shadow. Produce RS, RS, &c. to o, o, &c. and draw gh, gh, &c. perpendicular to Av, and respectively equal to GH, GH, &c. the conjugate axes of the several ellipses: then (since the solar rays may, so far as relates to this case, be reckoned parallel) will each of these ellipses be projected into parallel right lines, whose extremities are at gh, gh, &c. Now draw VOOO bisecting all the transverse axes, RS, RS, &c. which, as these axes are parallel to each other, will obviously be a right line. And since HG, HG, &c. are similar chords in circles whose dimensions increase as their respective distances VO, VO, &c. from the vertex they will be proportional to those distances: that is,

VO : VO :: HG : HG :: hg : hg. But by reason of the parallels Vv, Ro, &c. it is

VO : VO :: vo : vo; consequently vo : vo :: hg : hg.

And since this is the case in every parallel position of RGHS, and the corresponding projections hg, hg, &c. it is evident that they are the bases to a series of similar triangles, whose common vertex is v, and whose other sides are $vggg$, $vhhh$; that is, it is evident that $vggg$, $vhhh$; the lines bounding the shadow are right lines. But lines bounding the shadow, cannot, as is clear from the nature of such projections, pass *at a distance* from the base of the cone, as vI, vK; and they cannot, from what is done above, be other than right lines: therefore, *the lines bounding the shadow of a cone, standing in the sun-shine, are right lines drawn from the vertex of the shadow, and touching the base of the cone.* Q. E. D.

XX. QUESTION 320, *answered by Mr.* Cunliffe.

The construction of the question will be more neat if the following problem be first laid down as a Lemma.

In a given circle, whose centre is O, (fig. 688. pl. 33.) to draw a chord AB parallel to a right line OQ given by position, so that drawing OM perpendicular thereto, the sum of AB and MO may be of a given length or a maximum.

CONSTRUCTION. Take OP perpendicular to OQ, and equal to the given sum of the chord, and perpendicular thereon from the centre O; also take OQ = $\frac{1}{8}$OP, and draw PQ cutting the circle in B, B' then through either of the points B, B' draw the chord BA parallel to OQ, cutting OP in M, and the thing is done.

DEMON-

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DEMONSTRATION. Because OQ $= \frac{1}{2}$OP,
and OQ and MB are parallel,
therefore MB $= \frac{1}{2}$PM, or 2MB $=$ AB $=$ PM,
whence AB $+$ MO $=$ PM $+$ MO $=$ PO the given length
by the conſtruction.

Limit. When PO the ſum of the chord and perpendicular is a maximum, PQ will juſt touch the circle in B, and the radius OB will be perpendicular thereto by Euc. 18. III. hence and from what has been deduced PB $=$ 2OB from whence the line PB in the limiting caſe may be eaſily drawn.

The conſtruction of the original queſtion may be as follows:

The ſum of the ſquares of the ſides, by the queſtion, has a given ratio to the rectangle under the perpendicular and a given line; Wherefore it is pretty obvious, that the ſum of the ſquares of the ſides muſt be equal to the rectangle under the perpendicular and a *given line*, which given line let be denoted by 4L.

CONSTRUCTION. Take a right line O*d* (fig. 689. pl. 33.) equal to half the given difference of the ſegments of the baſe, and make the right angled triangle *d*OG ſuch that the hypothenuſe *d*G $=$ L; then with the centre O, and radius OG deſcribe a circle, in which by the preceding problem draw the chord BA parallel to O*d* ſo that AB $+$ MO may be a given length or a maximum.

Through *d* draw *d*D parallel to OM meeting AB in D, and in D*d* produced take *d*C $=$ L, join CA, CB and ACB is the required triangle.

DEMONST. Draw M*d*, MC and the radius OB.

Then $\overline{MB}^2 = \overline{OB}^2 - \overline{OM}^2 = \overline{OG}^2 - \overline{OM}^2$

$= \overline{dG}^2 - \overline{dM}^2 = \overline{dC}^2 - \overline{dM}^2$

by Euc. 47. I. and Simpſon's Geometry, theor. 9. b. 2.

Alſo $\overline{MC}^2 = \overline{dC}^2 + \overline{dM}^2 + 2dC \times dD$ by Euc. 12. II.

and adding the two foregoing expreſſions together,

$\overline{MD}^2 + \overline{MC}^2 = 2(dC)^2 + 2dC \times dD$

$= 2dC \times DC = 2L \times DC$: from whence and theorem 11. b. 2. Simpſon's Geometry,

$2(MB)^2 + 2(MC)^2 = \overline{AC}^2 + \overline{BC}^2 = 4L \times DC.$

Moreover BA $+$ D*d* is a given length, or a maximum by the conſtruction.

Therefore BA $+$ D*d* $+ dC =$ BA $+$ DC is a given length, or a maximum, becauſe *d*C is a given length.

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XXI. QUESTION 321, *anſwered by Mr.* Whitley.

Let ACB (fig 690. pl. 33.) be any plane triangle, AE, BF and CD lines drawn from the angles to biſect the oppoſite ſides, and interſecting each other in Q, and let QI, QG and QH be drawn perpendicular to the ſides AB, AC and BC. Then will the ſolid QI·QG QH be a maximum, and three times the ſum of the ſquares of AQ, BQ, CQ equal to the ſum of the ſquares of AC, BC, AB.

For draw FE, DF and DE, then becauſe CF = AF and CE = BE, CE : CF :: BE : AF :: CB : AC, therefore FE is parallel to AC. In the ſame way it is ſhewn that DF is parallel to BC and DE to AC.

Wherefore FE = DB = AD, DF = BE = CE, and DE = AF = CF.

Again, by ſimilar triangles FE : QE :: AB : AQ,
hence it follows that as AB is = 2EF,
AQ will be = 2QE, and therefore AQ = $\frac{2}{3}$AE.
In like manner it is ſhewn that
BQ = $\frac{2}{3}$BF, and CQ = $\frac{2}{3}$DC.

Wherefore $AQ^2 + BQ^2 + CQ^2 = (AE^2 + BF^2 + DC^2) \times \frac{4}{9}$

$= \left\{\begin{array}{l}\text{by Simpſon's Geometry II. 11, } (\frac{1}{2}AC^2 + \frac{1}{2}BC^2 - BE^2) \\ + (\frac{1}{2}AB^2 + \frac{1}{2}BC^2 - AF^2) + (\frac{1}{2}AC^2 + \frac{1}{2}BC^2 - AD^2)\end{array}\right\} \times \frac{4}{9}$

$= (AC^2 + BC^2 + AB^2 - BE^2 - AF^2 - AD^2) \times \frac{4}{9}$

$= (\frac{3}{4}AB^2 + \frac{3}{4}BC^2 + \frac{3}{4}AC^2) \times \frac{4}{9} = \frac{1}{3}AB^2 + \frac{1}{3}BC^2 + \frac{1}{3}AC^2.$

Conſequently $3AQ^2 + 3BQ^2 + 3CQ^2 = AC^2 + BC^2 + AB^2$.

Again, let CS be drawn perpendicular to AB;
then by ſimilar triangles CS : QI :: CD : QD;
but DQ = $\frac{1}{3}$CD (by above), theref. alſo QI = $\frac{1}{3}$CS.
Wherefore the △ AQB is equal to $\frac{1}{3}$ of the △ ACB.

In like manner it may be ſhewn that the triangles AQC, BQC are each equal to one-third of the triangle ACB, and conſequently the triangles AQB, AQC, and BQC, are equal to each other; therefore 2AQB × 2AQC × 2BQC is, by Simpſon's Geometry, Theorem 16, a maximum;

That is (DB·QI) × (FC·GQ) × (CE·QH) = a maximum, and becauſe DB, FC, CE, are here ſuppoſed given lines, the ſolid contained under QI × QG × QH is alſo a maximum. *Q. E. D.*

XXII.

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XXII. QUESTION 392, *answered by Mr.* W. Wallace.

It is evident that the centre of the given circle is such a point as satisfies the conditions of the question; but let us investigate whether another point can be found, such as is required.

ANALYSIS. Suppose Q (fig. 671, pl. 32.) to be the point which may be found and that DQ produced cuts the circle again in A, then by hypothesis the rectangle CQ × QD is given; but Q being a determinate point, from the nature of the circle AQ × QD is given, therefore the ratio of CQ × QD to AQ × QD is given and hence CQ must have to AQ a given ratio.

Join PQ cutting the circle in B and E; if we now suppose the point C to approach to B, the point A will also approach to B, so that when CQ coincides with BQ, AQ also coincides with BQ; thus it appears that the given ratio of CQ to AQ can be no other than that of equality, and since the line BQ must lie between AQ and CQ, however near these lines may be to each other, it is evident that BQ must pass through F the centre of the circle; thus the arch AB is equal to the arch BC, and if AF, CF be drawn to the centre, the angle AFB is equal to CFB therefore the angle AFC is double the angle AFB or AFP; but AFC being an angle at the centre is also double the angle ADC or ADP at the circumference, therefore the angles AFP, ADP are equal, hence the points A, F, D, B are in the circumference of a circle and the rectangle PQ × QF is equal to AQ × QD, that is to BQ × QF, to each of these rectangles add QF^2, and we have PF × FQ = BF^2. The point Q may therefore be found by the following CONSTRUCTION.

Let F be the centre of the given circle, and P the given point. Join PF and take the point Q between P and F so that FQ may be a third proportional to PF and FB the radius of the circle.

The same, by Mr. Cunliffe.

Through the centre O (fig. 692, pl. 33.) draw the right line PAB cutting the circle in A and B; then upon PO take PQ a fourth proportional to PO, PB, and PA, and Q is the point.

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DEMONSTRATION. Draw the radii CO, DO;
then by conſtruction PO : PB :: PA : PQ,
whence PO × PQ = PA × PB; but by Euc. 36. III. PA × PB = PC × PD, and conſequently PO × PQ = PC × PD, and therefore a circle will paſs through the points C, Q, O, D. Therefore ∠ PDQ = ∠ POC by Euc. 21. III, and conſequently ∠ PQD = ∠ PCO. Alſo ∠ CQO + ∠ PDO = ∠ CQO + ∠ OCD = ∠ PCO + ∠ OCD = 2 right angles by Euc. 13. I, and 22. III, and taking away ∠ OCD leaves ∠ CQO = ∠ PCO = ∠ PQD, and conſequently ∠ QCO = ∠ QPD; therefore the triangles CQO and PQD are ſimilar, therefore PQ : DQ :: CQ : QO, whence CQ × DQ = PQ × QO a given ſpace.

Mr. Whitley *anſwered it.*

XXIII. QUESTION 323, *anſwered by Mr.* Cunliffe.

Fig. 693. pl. 33. By a well known property of triangles $(AC - BC)^2 = 4DQ \times EF$, and therefore
the ſolid $(AC - BC)^2 \times DC = 4DQ \times DC \times EF$,
and when this ſolid is a maximum, the rectangle DC × EF will be a maximum, becauſe 4DQ is given; alſo when the rectangle DC × EF is a maximum, its ſquare $DC^2 \times EF^2$ muſt obviouſly be a maximum.
Draw FC; then by Euc. 13. II. $DC^2 + 2DF \times EF = DF^2 \times FC^2$, and by a well known property
FQ × EF = 2OF × EF = FC^2,
the difference of which is
$DC^2 + 2OD \times EF = DF^2 = 2OD \times L$, by taking the line L a third proportional to 2OD and DF; whence
$DC^2 = 2OD \times L - 2OD \times EF = 2OD \times (L - EF)$.
Therefore $DC^2 \times EF^2 = 2OD \times (L - EF) \times EF^2$,
and when this expreſſion is a maximum,
$(L - EF) \times EF^2$ will be a max. becauſe 2OD is given.
Now when $(L - EF) \times EF^2$ is a maximum, EF = ⅓L, as appears from Theorem 17th Simpſon on the maxima and minima; therefore 3EF = 2L.
Whence $3DO \times EF = 2DO \times L = DF^2$.

Q. E. D.

The

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The ſame, by Mr. Whitley, *of Atterciffe.*

Let fig. 694, pl. 33, be conſtructed as per queſt. and join AF, CF, AO, AQ, and to the point A draw the tange it AP meeting the diameter FQ in P, alſo biſect PD in S. Now becauſe the angles PAO, QAF are right angles (Euc. 31, 18, III.) the rectangles PD × DO, QD × DF are each equal to the ſquare of AD (Euc. Cor. 8, VI.)
therefore the rectangle PD × DO = QD × DF = 2DS × DO, add twice the rectangle DF × DO, to each,
then (QD + 2DO) × DF = (2DS + 2DF) × DO = 2FS × DO;
but QD + 2DO = DF, therefore DF² = 2FS × DO.

Now let GC be taken equal to the difference of the ſides AC, BC; then it will be manifeſt from what is ſhewn in Prop. 98. of Simſon's Data that
AF : AB :: CF : CG, and therefore
AF² : AB² :: CF² : CG² :: CF² × CD : CG² × CD.

Now ſince AB and QF are conſtant, by the hypotheſis, AF is ſo too, and therefore when CG² × CD is a maximum, CF² × CD will be ſo likewiſe; but (by Cor. Euc. 8, VI) the rectangle QF × EF = CF²; wherefore CF² × CD = QF × EF × CD, and, becauſe QF is conſtant, when the ſolid CF² × CD is a maximum, the rectangle EF × CD, and conſequently EF² × CD² = 2OD × ES × EF² (by Prop. VI. Dr. Stewart's General Theorems) will be ſo too; and therefore ſince 2OD is conſtant, the ſolid EF² × ES will be a maximum, which is the caſe when when EF = 2ES (vide Simpſon's Geometry, Theo. 17, Max. et Min.);
wherefore 3EF = CES = 2FS, and 3EF × OD = 2FS × OD;
but it has been ſhewn that DF² = 2FS × OD;
wherefore 3EF × OD = DF².

Q. E. D.

XXIV. QUESTION 324, *anſwered by Mr.* John Whitley.

Let NnmM (fig. 695, pl. 33.) be a conic ſection; and let the lines be drawn as per queſtion, alſo let PM, PN be drawn, the latter meeting AB in *v*, and join *nm*. Now by conics *nm* is parallel to AB. And by ſimilar triangles

vn

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$vn : vC :: nN : NM$, alſo $vC : vN :: nm : nN$, whereſ.
$vn : vN :: nm : NM :: Pn : PN$ (by ſim. triangles).

That is the ſtraight line PN is harmonically divided by the parallels nm, AB, in the points n, v; and therefore by Cor. 2. to Props. 71, 64, 38, Books 1, 2, and 3, of Emerſon's Conics, the ſtraight lines joining the points PA, PB, will be tangents to the Conic Sections at A and B.

The ſame, anſwered by Limenus, *the Propoſer.*

LEMMA 1. Let lines be drawn from the extremities of any ordinate to a conic ſection through a point in the diameter; and the line joining their interſections with the curve will be parallel to the ordinate.

LEMMA 2. Let MN (fig. 652, pl. 32) and mn be two parallel lines, and let Mm and Nn meet in P, and Mn and Nm in C, then PC being joined will biſect the parallels Mm and Nn.

Join PC to meet MN in D, and it follows from Lemma 1, that mn drawn will be parallel to MN, and therefore PC biſects MN by Lemma 2, and P is conſequently in the diameter appertaining to AB and MN. Alſo becauſe AC is parallel to MN and CD biſects MN in D, the lines AC, MC, DC, and NC are harmonicals; and they interſect PM in P, m, a, M, and PN in P, n, b, N; therefore Pa, and Pb are harmonical means between the ſegments PM, Pm, and PN, Pn. And therfore all other right lines drawn from the point P to AB will be harmonicals means between the ſegments intercepted by P and the curve (Maclaurin De Lin. Geom.). But PA meeting AB and the curve in the ſame point A, the other point of interſection of PA with the curve neceſſarily coincides with the firſt; and therefore PA is a tangent at A; and for the ſame reaſon PB touches the curve in B.

Mr. Cunliffe *alſo favoured us with an ingenious demonſtration of this Propoſition.*

XXV. QUESTION 325, *anſwered by Mr.* Cunliffe.

In fig. 667, pl. 32, take any two points B and a in the diameter EF of a ſemicircle and draw from thence the right lines BC, aC, to any point C in the circumference; draw the radius OC and demit CD perpendicular to EF; alſo upon EF take AD = aD

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aD and join AC. Then Ba is evidently the difference of the segments of the base made by the perpendicular CD in the triangle ACB.

Take the line L such that the

rectangle $OB \times L = OC^2 - OB \times Oa$, then

$OC^2 = OB \times (L + Oa)$; whence by Euc. 12. II.

$BC^2 = OC^2 + OB^2 + 2OB \times OD$

$= OB \times (L + Oa) + OB^2 + 2OB \times OD$, and

$aC^2 = AC^2 = OC^2 + Oa^2 + 2Oa \times OD$

$= OB \times (L + Oa) + Oa^2 + 2Oa \times OD$

$= OB \times L + OB \times Oa + Oa^2 + 2Oa \times OD.$

Now let S denote a space to which the square of BC has the of DB to Oa, that is

$OB : Oa :: BC^2 = OB \times (L + Oa) + OB^2 + 2OB \times OD$

$: Oa \times (L + Oa) + OB \times Oa + 2Oa \times OD$

$= Oa \times L + Oa^2 + OB \times Oa + 2Oa \times OD$

$= S$, whence

$AC^2 - S = OB \times L - Oa \times L = (OB - Oa) \times L$

$= Ba \times L$, and hence the following construction is derived.

Take the right line Ba equal to the given difference of the segments of the base and bisect the same in m, in Ba take OB to Oa in the given ratio mentioned in the question, and O will be a given point. Again take a right line L such that the rectangle $Ba \times L$ may be equal to the given excess of the square of one side above the space to which the square of the other side has a given ratio; then in OB take OF such, that $OF^2 = OB \times (L + Oa)$; with the centre O and radius OF describe a semicircle ECF the circumference of which will be the locus of the vertex C of the triangle. All therefore that now remains to be done is to draw the perpendicular DC so that the solid $AB \times DC^2 = 2mD \times DC^2$ may be a maximum, or that $mD \times DC^2$ may be a maximum, which is obviously the same thing as within the given segment of the sphere to inscribe the greatest cylinder possible; and the method of doing this is too well known to need repeating here; it belongs to the Scholium to Theo. 19, Simpson on the Maxima and Minima.

XXVI. QUESTION 326, *answered by the* Proposer.

From the nature of *converging* series, it is evident that in the form proposed, x must necessarily be always less than *unity*. And

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And it as evident that the coefficients, a, b, c, &c. muſt not only be in a regular decreaſing progreſſion, but in order that the terms of the ſeries may continue *ſine fine*, they muſt alſo be fractional numbers, or the reciprocals of an increaſing progreſſion of integers. Under theſe circumſtances (for in no other can the *infinite* ſummation be poſſible) we may, by the aſſiſtance of NEWTON's reſidual theorem, always obtain an expreſſion for the value of the propoſed infinite ſeries. Thus, if a, b, c, &c. be expounded by the reciprocals of the natural numbers 1, 2, 3, &c. the ſum of the ſeries reſulting from the application of the theorem adverted to, will be had from this ſimple *formula*.—*Subtract* x *from unity, divide unity by the remainder, from the* n*th root of the quotient take unity, multiply the remainder by* n, *and divide the product by* x, *the quotient will give the ſum of the ſeries* ad infinitum; when x is any large power of 10. In like mannaer may general expreſſions be reduced from the ſaid reſidual theorem with other values *ad libitum* of the coefficients a, b, c, &c. under the above limitations.

It is very remarkable that the above formula is obviouſly equal to *Napier's* logarithm of the reciprocal of $1 - x$ divided by x. Hence, for example if x be $\frac{9}{10}$, the ſum of the ſeries will be expreſſed by——the log. of 10 divided by ·9, equal to 2·558427. If x be $\frac{98}{99}$, the ſum will be obtained by dividing the log. of 99 by the compound repetend ·98, equal to 4·642; and thus with any other values of x than unity.

Cavallerius refers the propopoſer of this queſtion to pa. 222, 223, &c. Simpſon's Algebra, 2d edit. where he will find the method of ſumming the propoſed ſeries in a variety of caſes, when the law of the coefficients is given.

XXVII. QUESTION 327, *anſwered by Mr.* Cunliffe.

In the preſent caſe $y = b$, and the equation of the curve becomes $x = \frac{l^2 - y^2}{2y} \times \text{h. l.} \left(\frac{l + y}{l - y}\right)$,

and

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and taking the fluxions

$$\dot{x} = \frac{l\dot{y}}{y} - \dot{y} \times \left(\frac{l^2 + y^2}{2y^2}\right) \times \text{h. l.} \left(\frac{l+y}{l-y}\right), \text{ and hence}$$

$$y\dot{x} = l\dot{y} - \frac{\dot{y}}{2y} \times (l^2 + y^2) \times \text{h. l.} \left(\frac{l+y}{l-y}\right)$$

$$= l\dot{y} - \frac{y\dot{y}}{2} \times \text{h. l.} \left(\frac{l+y}{l-y}\right) - \frac{l^2\dot{y}}{2y} \times \text{h. l.} \left(\frac{l+y}{l-y}\right)$$

$$=$$ the fluxion of the area.

$$\text{And fluent of } y\dot{x} = \begin{cases} l\dot{y} - \text{fluent of } \frac{y\dot{y}}{2} \times \text{h. l.} \left(\frac{l+y}{l-y}\right) \\ - \text{fluent of } \frac{l^2\dot{y}}{2y} \times \text{h. l.} \left(\frac{l+y}{l-y}\right). \end{cases}$$

Now the fluent of $\frac{y\dot{y}}{2} \times \text{h. l.} \left(\frac{l+y}{l-y}\right)$ is

$$= \tfrac{1}{2}ly - \tfrac{1}{4} \times (l^2 - y^2) \times \text{h. l.} \left(\frac{l+y}{l-y}\right)$$

and the fluent of $\frac{l^2\dot{y}}{2y} \times \text{h. l.} \left(\frac{l+y}{l-y}\right)$ is

$$= l^2 \times \left(\frac{y}{l} + \frac{y^3}{9l^3} + \frac{y^5}{25l^5} + \frac{y^7}{49l^7} + \frac{y^9}{81l^9} + \text{&c.}\right)$$

$$\text{Whence the fluent of } y\dot{x} \left\{ = \begin{cases} l\dot{y} - \text{fluent of } \frac{y\dot{y}}{2} \times \text{h. l.} \left(\frac{l+y}{l-y}\right) \\ - \text{fluent of } \frac{l^2\dot{y}}{2y} \times \text{h. l.} \left(\frac{l+y}{l-y}\right) \end{cases} \right.$$

$$= \begin{cases} \frac{1}{2}ly + \frac{1}{4} \times (l^2 - y^2) \times \text{h. l.} \left(\frac{l+y}{l-y}\right) \\ - l^2 \times \left(\frac{y}{l} + \frac{y^3}{9l^3} + \frac{y^5}{25l^5} + \frac{y^7}{49l^7} + \frac{y^9}{81l^9} + \text{&c.}\right): \end{cases}$$

but this expreſſion ought to vaniſh when $y = l$, therefore the correct expreſſion for the area is

=

$$= \begin{cases} l^2 \times (1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} + \&c.) \\ -\frac{l}{2} + \frac{ly}{2} + \frac{1}{4} \times (l^2 - y^2) \times \text{h. l.} (\frac{l+y}{l-y}) \\ -l^2 \times (\frac{y}{l} + \frac{y^3}{9l^3} + \frac{y^5}{25l^5} + \frac{y^7}{49l^7} + \frac{y^9}{81l^9} + \&c. \end{cases}$$

$$= \begin{cases} \frac{l^2q^2}{2} - \frac{l^2}{2} + \frac{ly}{2} + \frac{1}{4} \times (l^2 - y^2) \times \text{h. l.} (\frac{l+y}{l-y}) \\ -l^2 \times (\frac{y}{l} + \frac{y^3}{9l^3} + \frac{y^5}{25l^5} + \frac{y^7}{49l^7} + \frac{y^9}{81l^9} + \&c.) \end{cases}$$

becauſe $\frac{l^2q^2}{2} = l^2 \ (1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81}$ &c.) as appears from Art. 6. of Landen's 5th Memoir; whence q denotes the length of the quadrantal arc of a circle whoſe radius is 1. And when $y = 0$, or when the chain is completely ſtretched along the horizontal line AB, the expreſſion for the area will then be

$$\frac{l^2}{2} \times (q^2 - 1) = l^2 \times .7337005.$$

The area may alſo be expreſſed in other particular caſes of the value of y, without ſeries from what is deduced at Art. 40. Poſtſcript to Landen's 5th Memoir.

The ſame, by Mr. John Surtees.

Let $c = 3.1416$, $z = \text{AV} =$ half the length of the chain, $x = \text{SV}$, and $y = \text{AS} = \frac{z^2 - x^2}{2x} \times \text{h. l.} (\frac{z+x}{z-x})$.

Then the fluxion of the area $= yx =$

$$\frac{z^2\dot{x}}{2x} \times \text{h. l.} (\frac{z+x}{z-x}) - \frac{x\dot{x}}{2} \times \text{h. l.} (\frac{z+x}{z-x}).$$

The

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The fluent of which is $\left\{ zx + \frac{x^3}{9z} + \frac{x^5}{25z^3} + \frac{x^7}{49z^5} \text{ &c.} \right\} + \frac{z^2 - x^2}{4} \times \text{h. l.} \left(\frac{z+x}{z-x}\right) - \frac{zx}{2}$.

And when $x = z$, the fluent is

$= z^2 \times (1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49})$ &c. $- \frac{z^2}{2} = z^2 \times \frac{c^2 - 4}{8}$
$= z^2 \times .73369 =$ the area.

W. W. R.

XXVIII. QUESTION 328, *answered by Mr.* I. H. Hearding

Put $d = 25$ inches, the distance of the axis of motion from the centre of the ball, and $r = 5$ inches, the ball's radius; then, by Emerson's Mechanics, Prop. 58. the distance of the centre of oscillation from the point of suspension, or the true length of the pendulum, is $= d + \frac{2rr}{5d} = 25\frac{1}{3}$ inches. Let AC (fig. 668, pl. 32.) represent the wire when stretched out tight in an horizontal position, AB its position when it first becomes perpendicular to the same; and, from any point D, in the arch CBH, wherein the vibrations are made, draw $Dd \parallel CH$; and let $AB = AC$ be put $= a$, $Bd = x$, and the arch $BD = z$; then, by the nature of the circle, $\dot{z} = \frac{a\dot{x}}{\sqrt{(2ax - xx)}}$; and, since the fluxion of the time through BD is as $\frac{\dot{x}}{\sqrt{[Ad]}}$ (see Art. 207, Simpson's Fluxions) it will be here truly expressed by

$$\frac{a\dot{x}}{\sqrt{[a-x]} \times \sqrt{[2ax-x^2]}} = \frac{\sqrt{[\frac{1}{2}a]} \times \dot{x}}{\sqrt{[ax-xx]}} \times (1 - \frac{x}{2a})^{-\frac{1}{2}}$$

$$= \frac{\sqrt{[\frac{1}{2}a]} \times \dot{x}}{\sqrt{[ax-xx]}} \times (1 + \frac{x}{2.2a} + \frac{3x^2}{2.4.4a^2} + \frac{3.5x^3}{2.4.6.8a^3} + \frac{3.5.7x^4}{2.4.6.8.16a^4}$$
&c.)

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and the fluent of this, when x becomes $= a$, is $=$

$$p\sqrt{[\tfrac{1}{2}a]} \times \left(1 + \frac{1}{2.2.2} + \frac{3\,3}{2.2.4.4.2^2} + \frac{3.3.5.5}{2.2.4.4.6.6.2^3} \text{ \&c.}\right)$$

where p is $= 3.14159$ &c. Now the time of descent through the diameter BF, or chord CB, is found to be as the fluent of

$$\frac{2a}{\sqrt{[2a]}}, \text{ or } 2\sqrt{[2a]};$$

therefore, this is to the time of descent through the arc CDB,

$$\text{as } \frac{4}{p} : 1 + \frac{1}{2.2.2} + \frac{3.3}{2.2.4.4.2^2} + \frac{3.3.5.5}{2.2.4.4.6.6.2^3} + \&.$$

$(= 1.180339)$.

But the absolute time of descent through the chord BC, or diameter BF is $= \sqrt{[BF \div 193]} = 0.51237$ seconds (Simpson's Select Exercises, pa. 183, 1st. Edit.); consequently as $1.27324 : 1.180339 :: 0.51237$ sec. $: 0.474986$ seconds, the time in which the wire will first become perpendicular to the horizon.

W. W. R.

The same, by Mr. John Smith, *Alton Park.*

By Dr. Hutton's Dictionary, Vol. I. pa. 268, the distance of the centre of oscillation from the point of suspension will be

$$25 + \frac{2 \times 5^2}{5 \times 25} = 25.4.$$

Then, by the nature of pendulums, as $\sqrt{39.125} : \sqrt{25.4} :: 5'' : .402865''$ the time of a semi-vibration in a cycloid. But by Dr. Hutton's Select Exercises, pa. 192, the time in a cycloid is to that in a circle, when the arc of semi-vibration is a quadrant, as 1 to 1.18014; therefore as $1 : 1.18014 :: .402865 : .475437$ of a second, the time required.

The same, by Mr. John Surtees.

Let $r = \frac{1}{12} \times \left(25 + \frac{2 \times 5^2}{5 \times 25}\right) = \frac{127}{60}$ feet the distance of the centre of oscillation from the point of suspension, $c =$

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$r\sqrt{[2]}$ = the chord of the arch which the centre of oſcillation deſcribes, $d = \frac{c}{4^{1.6}}$, and $s = 16\frac{1}{12}$ feet.

Then $ca = \frac{1}{2}$, and per Emerſon's Fluxions, page 145, the time of a ſemi-vibration

$$= \frac{3{\cdot}1416}{4}\sqrt{[\tfrac{2r}{s}]} \times : 1 + \frac{cd}{2^2} + \frac{3^2c^2d^2}{2^4\cdot 4^2} \text{ \&c.}$$

$$= \frac{3{\cdot}1416}{4}\sqrt{[\tfrac{254}{965}]} \times : 1 + \frac{1}{2^3} + \frac{1\cdot 3^2}{2^6\cdot 4^2} + \frac{1\cdot 3^2\cdot 5^2}{2^9\cdot 4^4\cdot 6^2} + \frac{1\cdot 3^2\cdot 5^2\cdot 7^2}{2^{12}\cdot 4^4\cdot 6^2\cdot 8^2} \text{ \&c.}$$

$= {\cdot}477$ ſeconds.

W. W. R.

XXIX. QUESTION 307, *anſwered by the* Propoſer.

Let x denote the required length of the lever AB, w equal 2lb. its weight per inch, and p ſuch a power applied at B, that may produce an equilibrium. Then from the principles of motion reſpecting the *heavy lever*, the momentum of the weight W, added to that of the lever itſelf, being equated to the momentum of the power acting at B, and properly tranſpoſed, it will then appear from the equation, that p is ſuſceptible of a *minimum*. Therefore, by the fluxionary proceſs *Maxima et Minima*, we find x equal to the ſquare root of AC·W divided by half of w, equal to 60 inches, the required length of the lever, when the power applied at B to raiſe the weight and beam will be the *leaſt poſſible*.

That is, $\text{AC}\cdot\text{W} + wx \times \frac{x}{2} = px$;

or, $\frac{\text{AC}\cdot\text{W}}{x} + \frac{wx}{2} = p = \text{AC}\cdot\text{W} \times x^{-1} + \frac{w}{2}x$;

hence $\frac{w}{2}\dot{x} = \text{AC}\cdot\text{W} \times x^{-2}\dot{x}$,

and $x = \sqrt{\frac{\text{AC}\cdot\text{W}}{\frac{1}{2}w}} = 60$.

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The ſame reſult may be obtained without the aid of fluxions, from the conſideration that when there is an equilibrium of power and weight in any mechanical combination, the motive force then applied will be only what is neceſſary to overcome the *vis inertiæ* and the friction of the parts.

Hence, from the common laws of motion, if the weight at C be ſo applied (by means of a pulley) as to raiſe the other end of the lever vertically, the relation of the parts when there is an equilibrium, will be expreſſed by an equation, which reduced, will give AB, the length of the lever, equal to 60 inches, as before.

From the above ſolution we ſee the propriety of the obſervation made by the ingenious Mechanician M. De Prony, in his valuable treatiſe *Architecture Hydraulique*, Art. 298, reſpecting the *Heavy Lever*, which ignorance or prejudice has laboured ſo ſtrenuouſly to repreſent as abſurd,——*viz.*

"*Ainſi lorſqu'on voudra ſoulever une maſſe* W *avec un levier peſant* AB, *il y aura une certaine longueur à donner à ce levier, afin que le moteur qu'on y applique ſoit le moindre poſſible.*"

The ſame, by Mr. John Surtees, *Houghton-le-Spring, Durham.*

Let z = the whole length of the lever in inches, then $2z$lbs. = its weight, and of conſequence half that weight will be ſupported at B, and $\frac{36 \times 100}{z}$lbs. = the part of the 100lbs. ſupported at B, which together $= z + \frac{3600}{z}$ = a minimum.

In fluxions $\dot{z} - \frac{3600\dot{z}}{z^2} = 0$. Hence $z = 60$ inches.

W. W. R.

The ſame, anſwered by Tyro Philomatheticus.

Let x = the length required, $a = 36$ inches, and w = weight of 100lbs. Then will aw = force of the weight, and $2x \times \frac{1}{2}x$

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$\frac{1}{2}x = x^2 =$ force of the lever; whence $\frac{aw + x^2}{x} =$ the power P, which, by the question is to be a minimum.

In fluxions $\frac{2x^2\dot{x} - aw\dot{x} - x^2\dot{x}}{x^2} = 0$, or $x^2 = aw$.

Hence $x = \sqrt{(aw)} = 60$ inches to length of the lever.

The same, by Mr. James Cunliffe.

If we imagine the lever AB, (fig. 669, pl. 32.) to be supported by the fulcra DA and EB in a position parallel to the horizon the weight w of 100lb. being appended at C 36 inches from A; then the question will be to determine the length of the lever AB so that the pressure sustained by the prop EB may be equal to the given weight p.

Put $AC = a$, and $AB = x$; then by the principles of mechanics, the part of the weight w sustained by the prop EB will be expressed by $\frac{AC \times w}{AB} = \frac{aw}{x}$. Also the part of the weight of the leaver AB sustained by the prop EB will be expressed by $\frac{AB \times s}{2} = \frac{sx}{2}$ where s denotes the weight of the lever per inch.

Therefore $\frac{aw}{x} + \frac{sx}{2} = \frac{2aw + sx^2}{2x}$ will express the whole weight sustained by the prop EB, which is to be equal to the given weight p by the question,
that is $2aw + sx^2 = 2px$, and this equation properly reduced gives $x = \frac{p}{s} + \frac{\sqrt{(p^2 - 2saw)}}{s}$.

Whence it is evident that p^2 must not be less than $2saw$; and when $p^2 = 2saw = 14400$ then the weight sustained by the prop EB will be the least possible, and in that case $x = \frac{p}{s} = .60$ inches the length of the lever AB.

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XXX. OR, PRIZE QUESTION 330, *anſwered by* Cavalerius.

Let BPC (fig. 700, pl. 33.) be the cycloidal curve, BO its axis, DC the baſe, BVD, MSC two ſemicircles (of the generating circle); and OS the line through the centres.

Draw EW, FX at equal diſtances from SO, and through K and L deſcribe two ſemicircles, having their centres in SO. Then from the geneſis of the curve, RP = PI; and becauſe SP = PV, SR will be = IV, viz. CH = GD.

Torricellius and others have demonſtrated that the trilineal ſpaces BMSCPB, BVDCPB are equal. Thus, becauſe BM = CD, WK = LN, SP = PV, XL = KV, &c. each figure conſiſts, or is made up, of an infinite number of parallel right lines, alike in each; and conſequently the areas muſt be equal.

Now in a ſimilar manner it is proved that the curve KPL biſects the mixt-lined quadrilateral KZLT; and that the trilineals KIP, LRP are equal; therefore the mixt-lined quadrilateral LRIT is = the trilinear. LKT. Hence the required cycloidal ſpace KLHG is equal to LRIT, = HG multiplied by the height DO; viz. equal to the diſtance of the centres of the circles multiplied by the radius.

This ſpace may be cut out of the circle two ways. Firſt, becauſe ZK, LT are equally diſtant from SO, the arc KT, or ZL, is equal to LT or HG, therefore the cycloidal ſpace is equal to the arc KT multiplied by the radius of the circle, or equal to a ſector having its arc double KT. Secondly, if the diſtance of ZK and LT be made equal to the radius, the mixt-lined quadrilateral ZLTK will be equal to the cycloidal ſpace, and may be cut out in that form.

The ſame, by the Propoſer, Marlovienſis.

Fig. 701, pl. 33. Let the cycloid be deſcribed as per queſt. and the generating circle in the ſituation propoſed, and alſo on the diameter BD, centre Z. At the interſection of the ordinates LK, FL, with the circumference of the generating circle, write

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F, Q; and join ZP, ZQ. Draw alſo the chords BP, BQ, DP, DQ, GK, HL. Then, from the well known properties of the circle and cycloid, we have

the cycloidal ſpace BLHD $\Big\}$ $= \Big\{$ BF × FL — circ. area BQF + rectilineal ſpace FLHD + circ. ſegm. HL.

And in like manner

the cycloidal ſpace BKGD is $\Big\}$ $= \Big\{$ BE × EK — circ. area BPE + rectilin. ſpace EKGD + circ. ſeg. GK;

the difference of which will evidently exhibit an expreſſion for the area of the cycloidal ſpace KLHG required. This method is general, wherever the points E, F are taken; but, when they are at equal diſtances from the centre Z, as in the queſtion, it becomes a particular caſe only, and the above general expreſſion reduces by expoſition to a very remarkable ſimple concluſion.

From the *Data*, and the properties of the figure, we have
BZ = ZD = ZP, EZ = ZF, EP = FQ,
arc BP = arc DQ = DG = PK, arc PQ = GH,
arc BQ = arc DP = arc BP + arc PQ,
BE = DF = BZ — EZ, BF = BZ + EZ,
EK = EP (FQ) + PK (DG) = EP + arc BP (arc DQ),
EL = EP (FQ) + arc BP (arc DQ) + arc PQ.

Call the area of the generating circle, A; the circular area DQF, or BPE, X; and the circ. ſegm. on HL, BP, or DQ, Y; and, for brevity, the propoſed cycloidal ſpace, S: And the circ. ſegm. on DP, BQ, or GK will be denoted by
$\frac{1}{4}$A — triang. BDQ — Y, or $\frac{1}{4}$A — BZ × EP — Y;
the circ. area BPQF by $\frac{1}{2}$A — X; and the rectilineal ſpaces FLHD, EKGD, reſpectively by

$$\frac{FQ}{2} \times FD + FD \times \overline{BP + \text{arc } PQ} =$$

$$\overline{BZ - EZ} \times \overline{\frac{FQ}{2} + BP + \text{arc } PQ}; \text{ and}$$

$$\frac{FQ}{2} \times \overline{BE + EZ} + \text{arc } BP \times BF.$$

Hence the former of the above expreſſions becomes

$$\overline{BZ + EZ} \times \overline{EP + \text{arc } BP + \text{arc } PQ} - \frac{A}{2} + X$$

+

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$+ \overline{BZ - EZ} \times \overline{\frac{1}{2}FQ + \text{arc } BP + \text{arc } PQ}$
$+ Y$; and the latter
$\overline{BZ - EZ} \times \overline{FQ + \text{arc } LP} - X$
$+ \overline{\frac{1}{2}FQ + \text{arc } LP\,(\text{arc } DQ)} \times \overline{BZ + EZ}$
$+ \frac{A}{2} - \overline{BZ \times FQ} - Y$;

which being expanded by multiplication, and reduced, become reſpectively.

$\frac{3}{2}BZ \times EP + 2BZ \times \text{arc } BP + \frac{1}{2}EZ \times EP - \frac{A}{2} + X$
$+ 2BZ \times \text{arc } PQ + Y =$ the cycloidal ſpace BLHD;
and $\frac{1}{2}BZ \times EP + 2BZ \times \text{arc } LP - \frac{1}{2}EZ \times EP$
$+ \frac{A}{2} - X - Y =$ the cycloidal ſpace BKGD.

The latter ſubtracted from the former, leaves
$BZ \times EP + EZ \times EP + 2X + 2Y + 2BZ \times \text{arc } PQ - A = S$.
But $BZ \cdot EP + 2Y$ is $=$ twice the ſector BZP,
ſo is $EZ \cdot EP + 2X =$ twice the ſame ſector;
and the quantity $2BZ \times$ arc PQ is evidently $= 4$ times the ſector ZPQ; therefore
$4 \times$ ſector BZP $+ 4 \times$ ſector ZPQ $- A = S$;
or, becauſe 4 ſector BZQ $= A +$ 2 ſector PZQ,
twice the ſector PZQ is $=$ the cycloidal ſpace KLHG.

Hence, ſince GH is equal the arc PQ, the conciſe theorem required is——*The product of the ſemidiameter of the generating circle into the diſtance* GH *is equal the cycloidal ſpace contained within the arcs* KG *and* LH; and the cutting off a ſpace from the generating circle equal to the cycloidal ſpace *geometrically* is obvious, being the circular ſector on twice the arc PQ.

Q. F. O.

The ſame, by Mr. Wallace, *of the Royal Military College.*

Fig. 702, pl. 33. Inſtead of ſuppoſing the points E and F equally diſtant from the centre, let us ſuppoſe them taken any where

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where in the axis. Let EK, FL, meet the generating circle in M and N. Draw the chords DM, DN, GK, HL; then from the nature of the curve DM, GK are parallel, alſo DN, HL. Now it was firſt ſhewn by Dr. Wallis (in his Treatiſe *De Motu* Cap. V. Prop. 20.) that if FL be any ordinate to the axis, meeting the generating circle in N, and LH be drawn parallel to the chord DN, meeting the baſe in H, the cycloidal area BKLHD is triple the area contained by the arch BMN, the axis BD, and the chord DN, to theſe let the equal ſegments H*p*L, DPN of the generating circle be added, and the mixtilineal area BKL*p*HD will be equal to half the generating circle together with twice the trilineal area BMND. For the ſame reaſon the mixtilineal area BK*q*GD is equal to half the generating circle together with the trilineal area BMD, therefore, taking the latter equation from the former, the propoſed cycloidal ſpace L*p*HG*q*K is double the area contained by the chords DM, DN and the arch MN. Hence the Theorem, as well as the geometrical method required in the queſtion is obvious, not only for that particular caſe, but for any given poſitions of the generating circle whatever.

Ingenious ſolutions to this queſtion were alſo received from Meſſrs. Cunliffe, Surtees, *and* Whitley.

The Medal for ſolving the Mathematical Prize Queſtion is decided in favour of Mr. JOHN SURTEES, who will pleaſe to ſend for it to Mr. GLENDINNING'S.

ARTICLE XXII.

On the Quadrature of the Circle.

By Mr. Benjamin Gompertz, *London.*

A certain Author attempting to ſquare the circle expreſſes himſelf to the following effect.

"If a tangent be drawn to a given circle, and from the point of contact a line be drawn through the centre and produced indefinitely on that ſide of the tangent where the centre of the circle is, and

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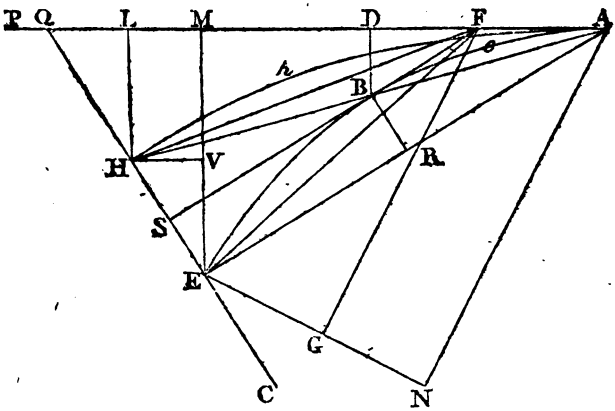
and there be taken an indefinite number of points in the lines thus produced beyond the centre and arcs of circles be described about every one of these as centres, through the said point of contact on one and the same side of the said line produced and each equal in length to the quadrant arc of the given circle reckoning from the aforesaid point of contact to points in the circular arcs, now described, that the locus of these points in the arcs will be a circle, and moreover that the centre thereof will be in the said tangent."

In order to determine the truth or fallacy of this I will premise the following Lemma.



LEMMA. Let PA be the aforesaid tangent, A the aforesaid point of contact, AH, AE the two lines passing through A and cutting the locus in H and E, the ∠ EAP being double HAP, or which is the same thing HAE = HAP, also let A*h*H, A*e*E be arcs of circles of equal length (and equal to the quadrantal arc of the given circle) passing through H, A, and E, A whose centres are on the same side of AP and in the line perpendicular thereto at A; join H, E and make HL perpendicular to AP; then I say HE is equal to HL and angle HEA a right angle.

For from the point B, the intersection of A*e*E and AH, draw BD and BR perpendicular to AP and AE, and join B, E; now since the centres of A*h*H and A*e*B are in the line perpendicular to AP at A, AP is a tangent to both, consequently the angle HAL, the angle of chord and tangent of both arcs, being the same

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ſame the arcs are ſimilar. Furthermore the angle AEB in the alternate ſegment is equal to the angle BAD, which by hypotheſis is equal the angle BAE, therefore BAE = BEA, therefore AR = AD = RE, conſequently 2AD = AE, and AeE, that is by hypotheſis, AhH = 2AeB, therefore is AH = 2AB, or by ſimilar △s AL = 2AD, equal from above to AE; hence the triangles AHL and AHE having AH common, the angles at A equal, and the ſide AL in the one equal to AE in the other, are equal in every reſpect; therefore HE = HL and ∠ HEA equal the right ∠ ALH. *Q. E. D.*

COROLLARY. The tangent to the locus at its interſection with AP is perpendicular to AP, and conſequently if the figure were a circle the centre would be in the line AP and therefore the latter part of the aſſertion would be true; for produce HE both ways to C and to Q, its interſection with AP, then the ∠ CQA is equal a right angle leſs the angle EAQ, but it is evident if H and E were to coincide CQ would be a tangent to the coincident points H and E, and then the angle EAH vaniſhing, EAQ its double muſt alſo vaniſh, and therefore E, H, Q coincide with the interſection of the locus and AP; alſo Q a right angle and CH a tangent to the locus at the ſaid interſection. *Q. E. D.*

Hence the locus cannot be a circle. For draw the right line AN making the angle NAP equal to 2EAP cutting the locus in N; join EN, biſect HE and EN in S and G; draw SF perpendicular to HE; join H, E, and G with F, and draw ME perpendicular and AP parallel to AP and ſuppoſe the locus a circle then its centre muſt be in the line SF biſecting the chord at right angles, but by the Corollary to the Lemma it is in AP, therefore F is the centre; conſequently the angle G of a line from the centre biſecting a chord a right angle, but by the Lemma ANE and AEH are right angles, therefore FS and FG parallel to AE and AN reſpectively; but by conſtruction, the angle NAP is double the angle EAP conſequently, ſubſtituting equals for equals, the angle GFP is double the angle SFP or the angle GFS equal to SFP, but SFE is equal to HFS, conſequently taking equals from equals, the angles EFG and HFP are equal, conſequently the triangles EFG and HFL having the angles at F equal, the angles at L and G equal, being both right angles, and the ſides HF and EF, being the radii of the ſame circle, equal; they are equal in every reſpect, and conſequently HL and EG equal, conſequently 2EG or EN and 2HL or HL more MV are equal, but by the Lemma EN is equal to EM or VE more MV, and HL equal to HE, conſequently VE and MV is equal to HE and MV, and

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and consequently VE and HE are equal, that is, the hypothenuse of a right angled triangle equal to one of the other sides which is absurd. *Q. E. D.*

Note. Let γ denote the radius of the given circle, q the length of a quadrant in the circle having unity for radius, and x the radius of one of the circles touching the given line at the given point of contact; then $\frac{\gamma}{x} \times q$ will denote the measure of the arch intercepted between the point of contact and the locus; and the length of the chord of the arch will be $2x \times \text{sin.} \frac{\gamma}{2x} \times q$.

But the angle made by the chord with the given tangent will be half the angle at the centre standing upon the arch; that is it will be measured by $\frac{\gamma}{2x} \times q$. Hence if we conceive the locus to be generated by the gyration of a straight line about the given point of contact as a pole, and call ρ the revolving radius and φ the measure of the angle made by it with the given tangent, we shall have

$$\rho = 2x \times \text{sin.} \frac{\gamma}{2x} \times q,$$

$$\varphi = \frac{\gamma}{2x} \times q.$$

And exterminating x, the equation of the curve is

$$\rho = \gamma \times \frac{q}{\varphi} \text{ sin. } \varphi.$$

It may be remarked that by taking x very great, or φ very small, the curve approaches to a circle of which $\gamma \times q$ is the radius, and the point of contact the centre, and that it ultimately coincides with this circle when x is infinite.

THE END.

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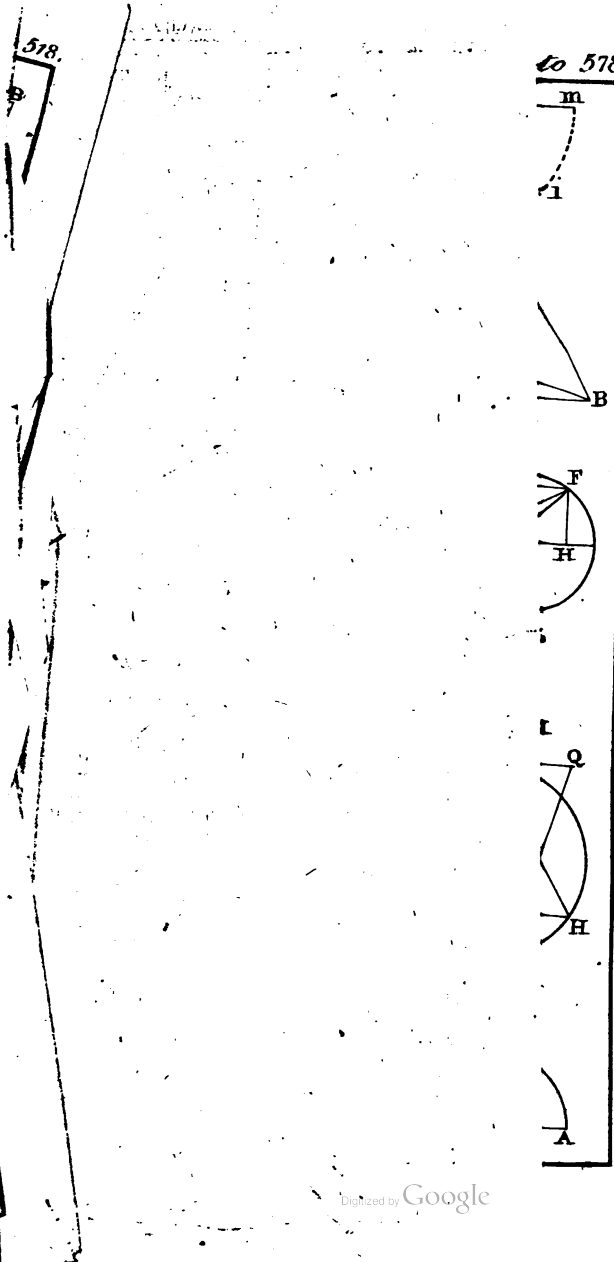

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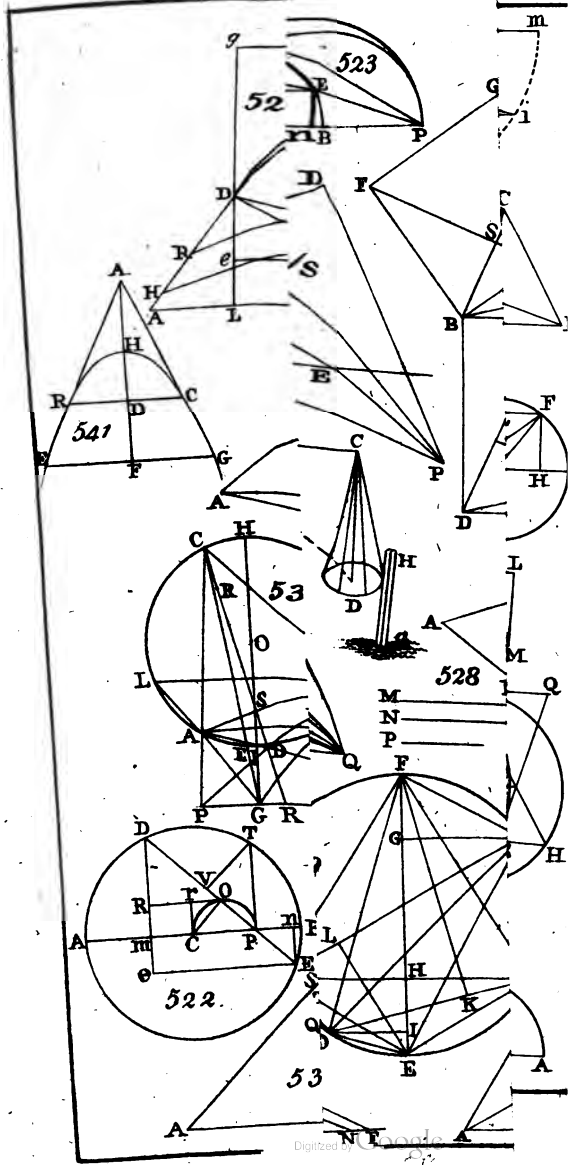

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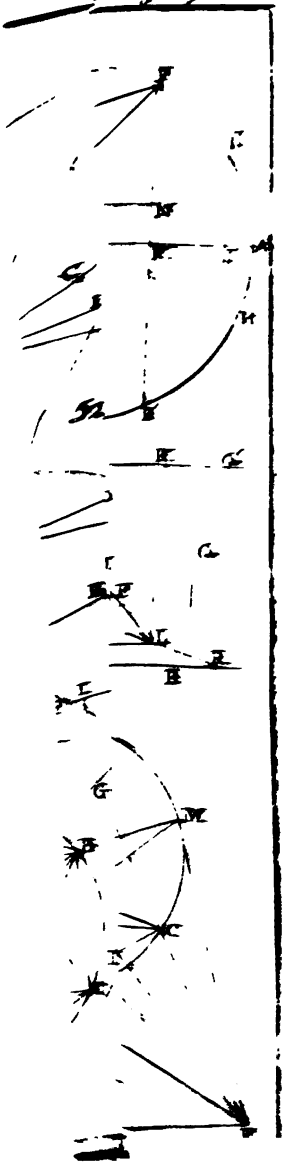

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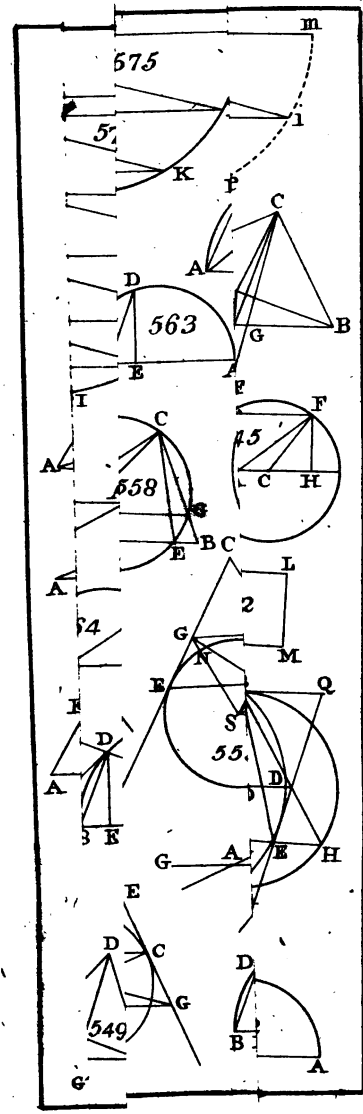

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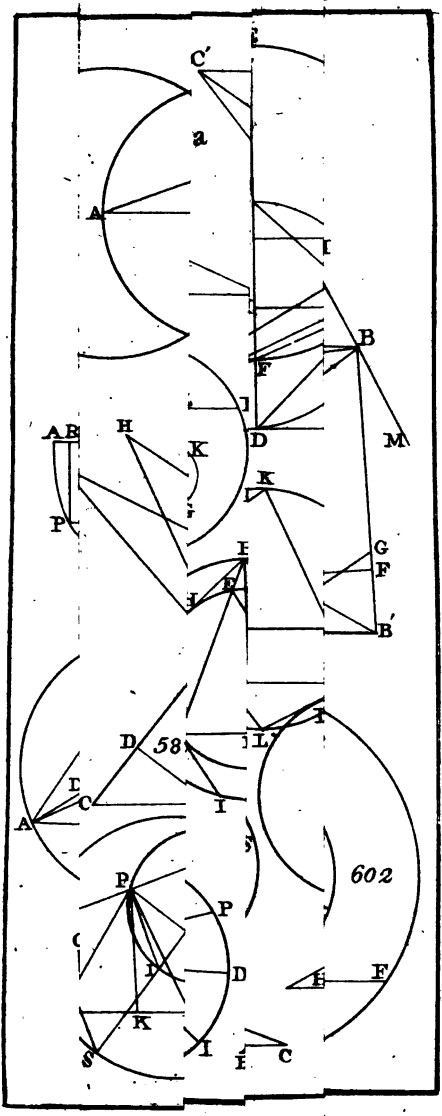

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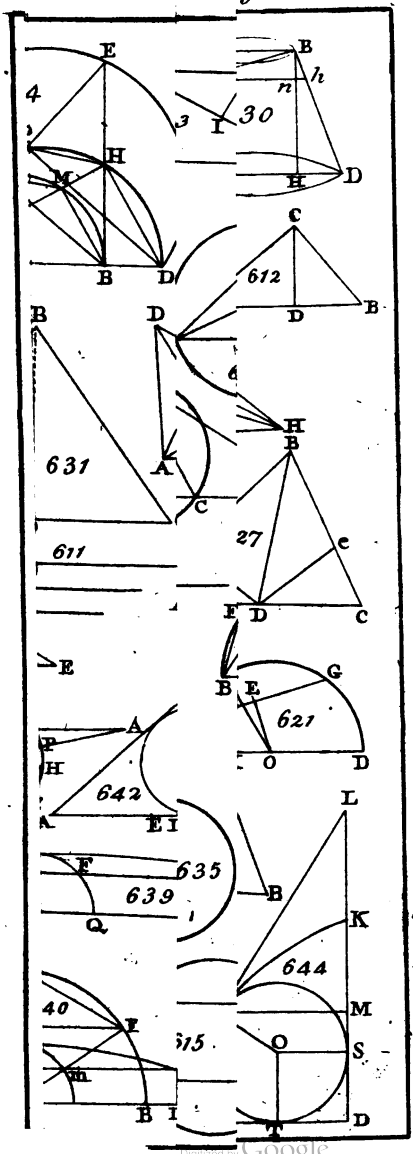

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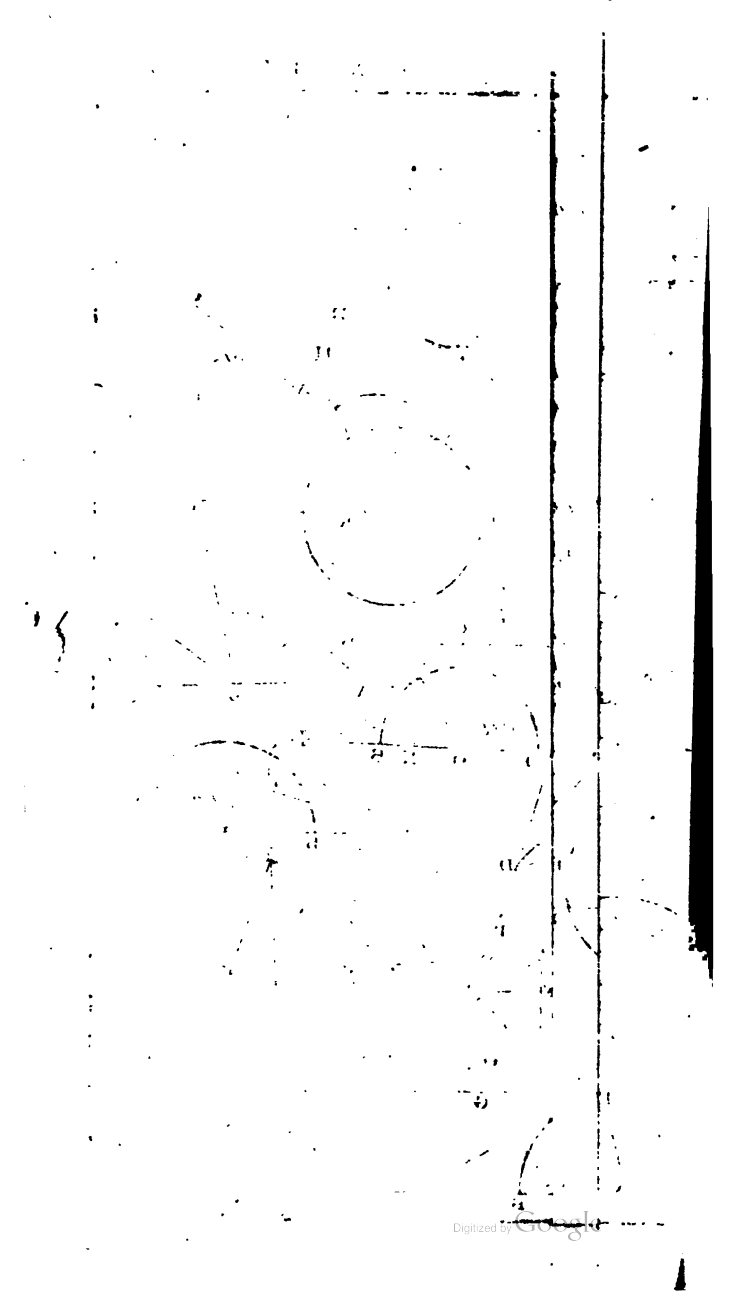
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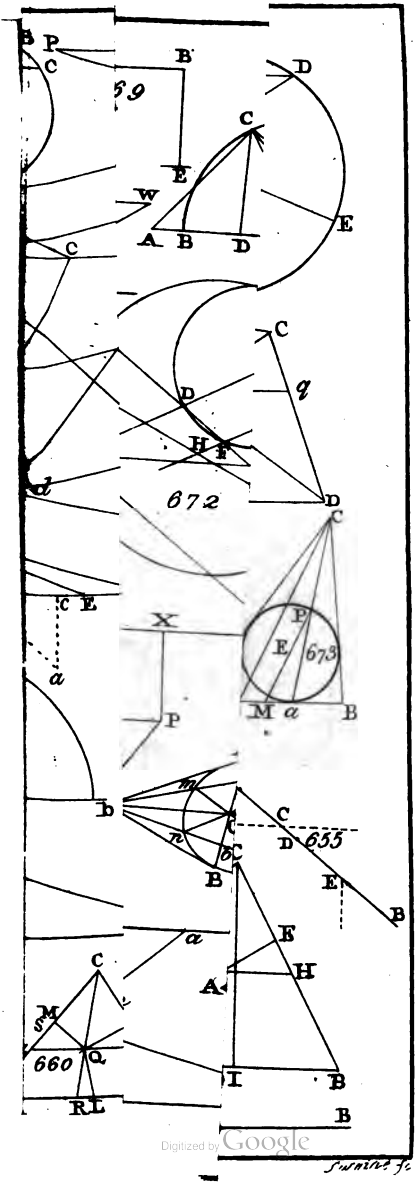

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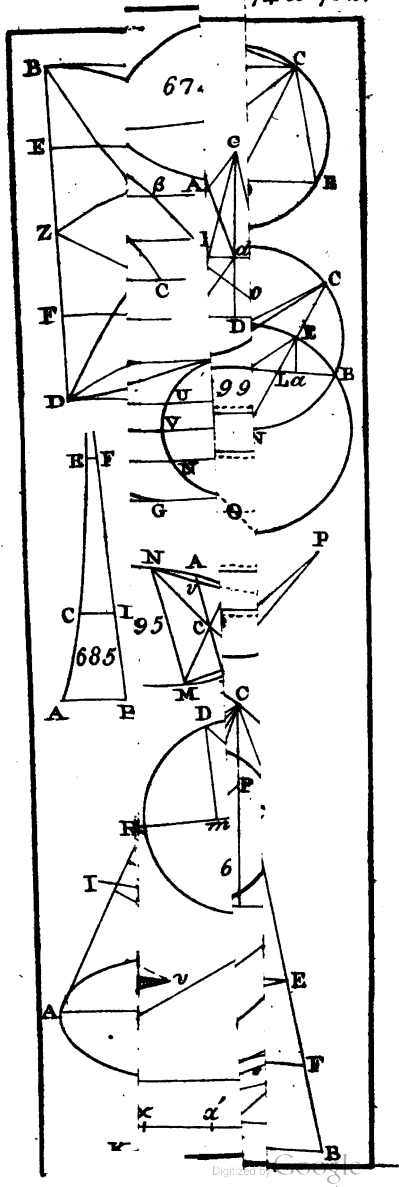
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